

Symmetry-Based Analysis and Conservation Laws of the Time-Fractional Hamilton–Jacobi–Bellman Equation with Exact Solutions

Hind Jawad Kadhum, Al-Bderi

University of AL Qadisiyah, Iraq

ABSTRACT

This paper presents an accurate application of the Lie symmetry method to find (and solve) the time-fractional Hamilton–Jacobi–Bellman (TFHJB) equation, which is a family of fractional differential equations that describe various nonlinear dynamical systems. The analysis commences with the detailed determination of the infinitesimal generators the equation possesses, which then serves as a basis for finding its respective Lie symmetries. Once these symmetries are recognized, the similarity reduction method will be applied to obtain exact solutions by reducing the original fractional equation into simpler cases. Additionally, the research goes beyond just determining the solutions and also includes deriving conservation laws associated with the TFHJB equation via a modified version of Ibragimov's method. These conservation laws yield fundamental information about the qualitative behavior, stability and integrability of the system.

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1. Introduction

Fractional differential equations (FDEs) have become increasingly popular in many fields of science and engineering in recent years. This is because these well-known models represent a wide range of intricate nonlinear processes as well as memory-related dynamics that are difficult for conventional integer-order differential equation techniques to adequately capture. Many processes in nature are influenced not only by the present moment in time but also by their historical behavior. An efficient mathematical foundation for describing such processes using integrals and derivatives of non-integer order is provided by fractional calculus. Due to this capability, it has become an important tool in several scientific disciplines, including physics, engineering, and mathematics. Fractional differential equations, in particular, are frequently employed to simulate systems with inherited traits and long-term memory.

There are various methods in the literature to solve FDEs (Alqhtani et al., 2023; Saad & Srivastava, 2023). The lack of well-established analytical methods has handicapped the study of fractional differential equations. However, Sophus Lie developed the Lie group transformation methodology, an analytical yet algorithmic method that offers a useful tool for obtaining an exact solution to a given differential equation. When a complicated system of PDEs or FDEs arises because of a chemically, physically, or physiologically significant issue, the discovery of any type of unambiguous solution is of great importance. Such explicit solutions may act as a model for physical investigations or as a check on numerical schemes. Often, these solutions represent the asymptotic or dominant features of the solution of a more general kind. Finding a set of differential equations' Lie symmetry group has several significant real-world uses. In addition, symmetry groups provide a useful tool for classifying different families of differential equations according to the presence of arbitrary parameters or functions. Invariance for ordinary differential equations with a single parameter We may lower the equation's order by one thanks to the Lie symmetry group. This makes it possible to obtain the solutions to the original issue from those of the simplified equation using just one quadrature. The form improves comprehension of the behavior of the solution by giving an explicit formula for the general solution to a first-order differential equation. Using conservation rules for standard differential equations, as was carried out in Lie group theory, is a second method for determining the solution to a first-order differential equation. A better towards the invariant properties of the equations can be arrived at by conservation laws. Furthermore, for some systems of differential equations, conservation rules are crucial in proving the existence, uniqueness, as well as stability of the solution, which helps us better understand the nature of the solution (Bluman &

Anco, 2002). To get correct solutions, Lie symmetry analysis of integro-order differential equations as well as linear differential equations has been explored in literature for a decade (Bluman & Kumei, 2013; Fushchich et al., 2013; Hashim et al., 2023; Ibragimov & Kovalev, 2009; Meleshko, 2006). The first extension of Lie theory to FDEs was done by Gazizov et al. (2007), while other researchers also conducted studies on FDEs and obtained favorable results (Djordjevic & Atanackovic, 2008; Wang et al., 2013).

For determining if a control is optimum with regard to a loss function, the HJBE, a nonlinear partial differential equation, offers both conditions that are sufficient and necessary. Decision theory and mathematical optimization both make use of loss functions, sometimes known as costs. Its objective is to map values or occurrences of one or more variables to a real number that reflects a "cost" associated with the occurrence. Minimizing the loss function is the aim of an optimization task. The optimal control problem's value function is provided by the HJBE solution. Finding the control that maximizes or minimizes the Hamiltonian function in the HJB equation can yield the ideal control approach after the value function has been determined.

The defined contribution pension plan has gained popularity in the global pension market. The benefits of this plan are contingent upon the results on the fund's portfolio and are based on fixed payments. Under this kind of plan, the employee is in charge of accumulating enough money to meet their retirement needs and assumes the investment risk. A single hazardous asset, S_t , and a risk-free asset, B_t , comprise the financial market taken into consideration in this paradigm.

$$B_t = rB_t dt,$$

$$\frac{dS_t}{S_t} = \mu dt + kS_t^\beta dW_t$$

Considering a "constant rate of interest r , an elasticity parameter $\beta < 0$, and an expected rate of return in real time $\mu > r$." It is assumed that the contribution rate for one unit of income is a constant $c \in (0, 1)$. Here, kS_t^β is the instantaneous volatility and W_t a standard Brownian motion. It is now possible to simulate the evolution of pension wealth V_t using the following formula:

$$dV_t = [\pi_t(\mu - r)V_t + rV_t + c]dt + \pi_t V_t kS_t^\beta dW_t$$

where T is the fund investment horizon and π_t is the portion of the pension fund that is devoted to riskier assets (Sun et al., 2020). The value function, as per the traditional stochastic optimum control theory,

$$H(s, t, v) = \sup_{\{\pi_t\}} E[U(V_T) | S_t = s, V_t = v] \quad , \quad 0 < t < T \quad (1)$$

with strictly concave utility function $U(VT)$, should satisfy the (HJ B) equation

$$H_t + \mu s H_s + (rv + c)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} - \frac{[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}]^2}{2k^2 s^{2\beta} H_{vv}} = 0$$

Process modelling is one of the core prerequisites for optimal control theory. The better the control, the more accurate the model. Several mathematical methods have been developed to simulate the process to be regulated in tandem with advancements in optimal control theory. Fractional calculus is one of these instruments. Fractional calculus has impacted the field of control engineering in two ways: by creating a strong framework for the closed loop control system and by offering better models for the processes. On the other hand, we have been interested in solving financial mathematics-related problems utilizing the mathematical theory of Lie algebras (Razminia et al., 2018). To our knowledge, however, lie group theory has not yet fully investigated the fractional order form of the well-known (HJB) equation. This study mainly concerns with the determining conservation laws, similarity reductions and Lie point symmetries of (HJB) type equations (TF). This is analysed as the time fractional HJB equation:

$$D_t^\alpha H + \mu s H_s + (rv + c)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} - \frac{[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}]^2}{2k^2 s^{2\beta} H_{vv}} = 0 \quad (2)$$

Where D_t^α In the Riemann-Liouville notion, is the fractional derivative for $0 < \alpha < 1$.

2. Preliminaries and Some Definitions

The concepts and theorems of fractional calculus that are necessary for our further progress are explained in this part. Refer to (Kilbas et al., 2006; Podlubny, 1999).

Definition 2.1. For an integrable function $f(t) \in (L1[a, b], R)$, the left-hand and right-hand Riemann-Liouville fractional integrals of order $\alpha > 0$ is written as follows:

$${}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a \quad (3)$$

$${}_x I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t < b, \quad (4)$$

where the Gamma function is represented by Γ (Podlubny, 1999).

The following are some characteristics of the operator I^α . For $\alpha, \gamma \geq 0$ and $\beta \geq -1$ we have:

$$I_t^\alpha I_t^\beta f(t) = I_t^{\alpha+\gamma} f(t), I_t^0 f(t) = f(t), I_t^\alpha I_t^\gamma f(t) = I_t^\gamma I_t^\alpha f(t), \quad I_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$$

Definition 2.2 Assume the definition of $f(t)$ is on $[a, b]$. The Riemann-Liouville derivatives (RLD) of order n are specified to be both left-sided and right-sided in order to describe the fractal behavior of a function.

$${}_a D_t^\alpha f(t) = D_t^n ({}_a I_t^{n-\alpha} f(t)) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (5)$$

$${}_t D_b^\alpha f(t) = (-1)^n D_t^n ({}_t I_b^{n-\alpha} f(t)) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau \quad (6)$$

where $n = [\alpha] + 1$.

A few properties of the RLD are given below. For $\alpha, \beta, \gamma > 0$ we have:

- $D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}$, where t^β is α differentiable
- $D_t^\gamma C = \frac{ct^{-\gamma}}{\Gamma(1-\gamma)}$, C is a constant,
- $D_t^\beta [h(t)l(t)] = \sum_{m=0}^{\infty} \binom{\beta}{m} D_t^{\beta-m} h(t) D_t^m l(t)$, $\beta > 0$.

The generalized Leibnitz rule is the last characteristic, where $\binom{\beta}{m} = \frac{(-1)^{m-1} \beta \Gamma(m-\beta)}{\Gamma(1-\beta) \Gamma(m+1)}$.

3. Analysis of Time Fractional Hamilton-Jacobi-Bellman Equations using Symmetry Methods

Applying Lie symmetry analysis to FDEs, we analyse and classifying the symmetries of the time-fractional Hamilton-Jacobi-Bellman equation (TFHJBE), in order to obtain invariant structures, investigate reduction and find solution-generating transformations. The Lie point symmetries of FDEs can be determined using the methodology outlined below. We start with infinitesimal starting point of Lie point transformation group, defined with single continuous parameter, which serves as stepping stone for the following analysis regarding symmetry. Then the associated infinitesimal generator of these transformations is provided. The consequent overdetermined system of PDEs and FDEs is obtained by applying the suitable prolongation (according to derivatives and fractional derivatives involved in the FDE) of this vector field to the corresponding FDE while maintaining the invariance. The Lie point symmetries of the associated FDE (Bluman & Kumei, 2013; Djordjevic & Atanackovic, 2008) can be obtained by calculating or utilizing the symmetry program FracSym (Jefferson & Carminati, 2014) to solve this overdetermined system.

Consider the (TFHJBE) in the form

$$F = D_t^\alpha H + \mu_s H_s + (rv + c)H_s + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} - \frac{[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}]^2}{2k^2 s^{2\beta} H_{vv}} = 0 \quad (7)$$

$$\bar{t} = t + \epsilon \tau(t, s, v, H) + o(\epsilon^2)$$

$$\bar{s} = s + \epsilon \xi(t, s, v, H) + o(\epsilon^2)$$

$$\bar{v} = v + \epsilon \eta(t, s, v, H) + o(\epsilon^2)$$

$$\bar{H} = H + \epsilon \zeta(t, s, v, H) + o(\epsilon^2) \quad (8)$$

We refer to the collection of all these transformations as a continuous Lie group G if the structure of equation (7) is maintained in the new variables $\bar{t}, \bar{s}, \bar{v}$ and $\bar{H}$. In the aforementioned transformations, ϵ is a continuous parameter. The equivalent infinitesimal generator for these transformations is given by:

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial s} + \eta \frac{\partial}{\partial v} + \zeta \frac{\partial}{\partial H} \quad (9)$$

$$\left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon=0} = \tau(t, s, v, H), \quad \left. \frac{d\bar{s}}{d\epsilon} \right|_{\epsilon=0} = \xi(t, s, v, H)$$

$$\left. \frac{d\bar{v}}{d\epsilon} \right|_{\epsilon=0} = \eta(t, s, v, H), \quad \left. \frac{d\bar{v}}{d\epsilon} \right|_{\epsilon=0} = \zeta(t, s, v, H) \quad (10)$$

Furthermore, in order to calculate the lower limit of the integral in (5) (Ibragimov, 2007), the Riemann-Liouville fractional derivative must preserve a defined form invariant under transformation (8), which is why we require that $t=0$ stay invariant under Lie point transformations. Therefore

$$\tau(t, s, v, H) \Big|_{t=0} = 0 \quad (11)$$

Because of the invariance criteria, only when the extended vector field $\text{Pr}^{(\alpha,t)} X$ vanishes on the results of Eq. (7) does Eq. (7) accept the transformation group (8) and the vector fields (9). In other words,

$$\text{Pr}^{(\alpha,t)} X(F) \Big|_{F=0} = 0 \quad (12)$$

This equation defines all infinitesimal symmetries of Eq. (7); therefore, we call that determining equation. Considering only the necessary terms, the extended operator $\text{Pr}^{(\alpha,t)} X$ is expressed as follows.

$$\text{Pr}^{(\alpha,t)} X = X + \zeta^{(\alpha,t)} \partial_{\alpha H} + \zeta^s \partial_s + \zeta^v \partial_v + \zeta^{ss} \partial_{ss} + \zeta^{vs} \partial_{vs} + \zeta^{vv} \partial_{vv} \quad (13)$$

To determine ζ^s, ζ^{ss} we concentrate on the following formulae (Bluman & Kumei, 2013; Ibragimov & Kovalev, 2009):

$$\begin{aligned} \zeta^s &= D_s(\zeta) - H_t D_s(\tau) - H_s D_s(\zeta) - H_v D_s(\eta) \\ &= \zeta_s + \zeta_H H_s - \tau_s H_t - \tau_H H_t H_s - \xi_s H_s - \xi_H H_s^2 - \eta_s H_v - \eta_H H_v H_s \\ \zeta^{ss} &= D_s(\zeta^s) - H_{ts} D_s(\tau) - H_{ss} D_s(\xi) - H_{vs} D_s(\eta) \\ &= \zeta_{ss} + 2\zeta_{sH} H_s + \zeta_{HH} H_s^2 + \zeta_H H_{ss} - \tau_{ss} H_t - 2\tau_{sH} H_t H_s - 2\tau_s H_{st} - \tau_{HH} H_t H_s^2 \\ &\quad - \tau_H H_t H_{ss} - 2\tau_H H_s H_{st} - \xi_{ss} H_s - 2\xi_{sH} H_s^2 - 2\xi_s H_{ss} - \xi_{HH} H_s^3 - 3\xi_H H_s H_{ss} \\ &\quad - \eta_{ss} H_v - 2\eta_{sH} H_v H_s - 2\eta_s H_{sv} - \eta_{HH} H_v H_s^2 - \eta_H H_v H_{ss} - 2\eta_H H_s H_{sv} \end{aligned} \quad (14)$$

The expressions for ζ^v, ζ^{vv} and ζ^{vs} are computed similarly. To determine the prolonged vector field (13), The fractional time derivative's α -th extended infinitesimal $\zeta^{(\alpha,t)}$ must be extracted using the following formula:

$$\begin{aligned} \zeta^{(\alpha,t)} &= D_t^\alpha(\zeta) + D_t^\alpha(\xi H_s) + \xi D_t^\alpha(H_s) + D_t^\alpha(H D_t(\tau)) - D_t^{\alpha+1}(\tau H) \\ &\quad + \tau D_t^{\alpha+1}(H) - D_t^\alpha(\eta H_v) + \xi D_t^\alpha(H_s) \end{aligned} \quad (15)$$

The chain rule and expanded Leibnitz rule provide the extended infinitesimal $\zeta^{(\alpha,t)}$ (see (Gazizov et al., 2007; Singla & Gupta, 2017)).

$$\begin{aligned} \zeta^{(\alpha,t)} &= \sum_{m=1}^{\infty} \left[\binom{\alpha}{m} \partial_t^m \zeta_H - \binom{\alpha}{m+1} D_t^{m+1}(\tau) \right] \partial_t^{\alpha-m} H + (\xi_H - \alpha D_t(\tau)) \partial_t^\alpha H - \sum_{m=1}^{\infty} \binom{\alpha}{m} D_t^m(\xi) \partial_t^{\alpha-m}(H_s) - \\ &\quad \sum_{m=1}^{\infty} \binom{\alpha}{m} D_t^m(\eta) \partial_t^{\alpha-m}(H_v) + \partial_t^\alpha \zeta - H \partial_t^\alpha \zeta_H + \mu \end{aligned} \quad (16)$$

$$\mu = \sum_{m=2}^{\infty} \sum_{n=2}^m \sum_{r=2}^n \sum_{s=0}^{r-1} \binom{\alpha}{m} \binom{m}{n} \frac{t^{m-\alpha}}{r! \Gamma(m+1-\alpha)} (-H)^s \frac{\partial^n}{\partial t^n} (H^{r-s}) \frac{\partial^{m-n+r} \zeta}{\partial t^{m-n} \partial H^r}$$

When the infinitesimal ζ is linear in H , we should remember that $\mu = 0$. Consequently, the time fractional (HJB) equation's invariance criteria (12) are suggested as

$$\begin{aligned} &\left[\zeta^{(\alpha,t)} + \mu s \zeta^s + \mu \xi H_s + (rv + c) \zeta^v + r \eta H_v + \frac{1}{2} k^2 s^{2\beta+2} \zeta^{ss} + \frac{1}{2} k^2 (2\beta+2) s^{2\beta+1} \xi H_{ss} + \frac{(\mu-r)^2 H_v \zeta^v}{k^2 s^{2\beta} H_{vv}} - \frac{(\mu-r)^2 H_v^2 \beta \xi}{k^2 s^{2\beta+1} H_{vv}} \right. \\ &\quad \frac{(\mu-r)^2 H_v^2 \zeta^{vv}}{2k^2 s^{2\beta} H_{vv}^2} + \frac{k^4 s^{4\beta+2} H_{vs} \zeta^{vs}}{k^2 s^{2\beta} H_{vv}} + \frac{(4\beta+2) k^4 s^{4\beta+1} H_{vs} \xi}{k^2 s^{2\beta} H_{vv}} - \frac{k^4 s^{4\beta+2} H_{vs}^2 \xi}{2k^2 s^{2\beta+1} H_{vv}} - \frac{k^4 s^{4\beta+2} H_{vs}^2 \zeta^{vv}}{2k^2 s^{2\beta} H_{vv}^2} + \frac{(\mu-r) k^2 s^{2\beta+2} H_{vs} \zeta^v}{k^2 s^{2\beta} H_{vv}} \\ &\quad \left. + \frac{(4\beta+2)(\mu-r) H_v k^2 s^{2\beta+1} H_{vs} \xi}{2k^2 s^{2\beta} H_{vv}} + \frac{(\mu-r) k^2 s^{2\beta+1} H_v \zeta^{vs}}{k^2 s^{2\beta} H_{vv}} - \frac{2(\mu-r) k^2 s^{2\beta+1} H_v H_{vs} \xi}{k^2 s^{2\beta+1} H_{vv}} - \frac{(\mu-r) k^2 s^{2\beta+1} H_v H_{vs} \zeta^{vv}}{k^2 s^{2\beta} H_{vv}^2} \right]_{(7)=0} = 0 \end{aligned} \quad (17)$$

By equating the monomial coefficients in partial derivatives of t, s and v using the prolongation formulas, the over determined system of linear ODEs and FODEs is obtained. After solve the system of these equations, the final form of infinitesimals can be derived:

$$\tau = C_1, \quad \xi = 0, \quad \eta = (c + rv) C_2 + C_3 s, \quad \zeta = F(t, s) + C_4 H \quad (18)$$

where c_i ($i = 1, 2, 3, 4$) and $F(t, s)$ are arbitrary constants and arbitrary function respectively (Razminia et al., 2018). By regarding the condition $\tau(0) = 0$, every admitted operator of equation (2) can be presented as:

$$X_1 = (c + rv) \frac{\partial}{\partial v}, \quad X_2 = s \frac{\partial}{\partial v}, \quad X_3 = H \frac{\partial}{\partial H}, \quad X_4 = F(t, s) \frac{\partial}{\partial H} \quad (19)$$

thus, Dimension of Lie algebra, admitted by equation fractional (HJB) (0.2) is $L_3 \oplus L_\infty$. We can compute the symmetries using the MAPLE symbolic computing platform since the process of obtaining symmetries for FDEs and lengthy computations is algorithmic. When used with the MAPLE symmetry packages DES-OLVII (Vu et al., 2012) & ASP (Butcher et al., 2003), the symmetry package FracSym (Jefferson & Carminati, 2014) can assist in identifying the infinitesimal generators and finding equations for the symmetries of FDEs.

4. Construction of Exact Solutions using Similarity Reductions

In this section, the similarity reduction technique is described and extended to obtain exact solutions of Eq. (2). The primary aim of this approach is to bring down partial differential equations into smaller PDEs with reduced count of independent variables or convert them to ordinary differential equations. If a function H is unaffected by the action of the infinitesimal generators connected to the equation's symmetries, or if the outcome of applying the generators to H is zero, it is referred to as an invariant solution (see (Djordjevic & Atanackovic, 2008; Singla & Gupta, 2017; Wang et al., 2013)). Using these symmetry properties, we derive group-invariant solutions for the fractional Eq. (2).

Case 1: For the symmetry $V_1 = X_1 + X_3$, by integrating the invariant surface

$$\frac{ds}{0} = \frac{dt}{0} = \frac{dv}{c+rv} = \frac{dH}{H} \quad (20)$$

Therefore, the similarity variables are t, s and $(c + rv)^{-\frac{1}{r}}$. Consequently, the invariant solution may be expressed as:

$$H = \sqrt[r]{c + rv} g(t, s) \quad (21)$$

where $g(t, s)$ satisfies the equation

$$\sqrt[r]{c + rv} \left(g_t^\alpha + g_s + g + \frac{1}{2} k^2 s^{2\beta+2} g_{ss} \right) - \frac{\left[\sqrt[r]{(c + rv)^{(1-r)}} \left((\mu - r)g + k^2 s^{2\beta+1} g_s \right) \right]^2}{2k^2 s^{2\beta} \sqrt[r]{(c + rv)^{(1-2r)}} (1 - r)g} = 0$$

Case 2: Consider generator $V_2 = X_2 + X_3$, invariant surface in this case is:

$$\frac{ds}{0} = \frac{dt}{0} = \frac{dv}{s} = \frac{dH}{H} \quad (22)$$

gives the similarity variable t, s and $H e^{-\frac{v}{s}}$. Thus Eq. (2) has the solution as

$$H = e^{\frac{v}{s}} f(t, s) \quad (23)$$

where $f(t, s)$ satisfies the equation

$$f_t^\alpha + \mu s f_s + (rv + c) \frac{1}{s} e^{\frac{v}{s}} f + \frac{1}{2} k^2 s^{2\beta+2} \left(\left(\frac{2v}{s^3} + \frac{v^2}{s^4} \right) e^{\frac{v}{s}} f - \frac{2v}{s^2} e^{\frac{v}{s}} f_s + e^{\frac{v}{s}} f_{ss} \right) - \frac{\left[(\mu - r) \frac{1}{s} e^{\frac{v}{s}} f + k^2 s^{2\beta+1} e^{\frac{v}{s}} \left(\frac{-(v+s)}{s^3} f + \frac{1}{s} f_s \right) \right]^2}{2k^2 s^{2\beta} e^{\frac{v}{s}} f} = 0$$

5. Conservation Laws Related to the Hamilton-Jacobi-Bellman Equation with Time Fraction

A basic concept states that during the evolution of an isolated system; the overall amount of a certain physical quantity stays constant. In mathematical terms, this principle is expressed as a conservation law. Conservation laws have a wide range of applications across various scientific disciplines. One of the most important uses of symmetry groups is in constructing singular solutions for PDEs, integro-differential equations, and their respective systems. It was Noether who first formally established the relation between symmetry groups and conservation laws in 1918, proving that, given a Lagrangian exists for the Euler-Lagrange equations, one could systematically determine the related conservation laws. The Noether's theorem offers a way to identify conservation laws. For systems with fractional derivatives, there is some literature that computes conservation laws by Noether's theorem (see (Agrawal, 2002) and (Riewe, 1996)). Based on this they also established conservation laws for equations describing fractional diffusion and the Noether's operators of their fractional extension in formal Lagrangian (Lukashchuk, 2015) and through Ibragimov classfractor theorem (Ibragimov, 2007). In this part, we derive the conservation rules for the TFHJB issue using Ibragimov's technique. For the fractional HJB equation (2) there is, as for any conservation law, an associated continuity equation that embodies this property

(Jabber & Eswad, 2025).

$$D_t(C^t) + D_s(C^s) + D_v(C^v) = 0 \Big|_{\text{Eq. (2)}=0} \quad (24)$$

where t, s, v, H , and derivatives of H are functions of the conserved vectors. Here D_i indicates the total derivative of variable i . Thus, the fractional equation (2)'s formal Lagrangian can be found as:

$$\mathcal{L} = V(t, s, v, H)F_j = V(t, s, v)(D_t^\alpha H + \mu s H_s + (rv + c)H_v + \frac{1}{2}k^2 s^{2\beta+2} H_{ss} - \frac{[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}]^2}{2k^2 s^{2\beta} H_{vv}}) \quad (25)$$

$V(t, s, v, H)$ represents the new dependent variable. This leads to the following form for the adjoint equation of equation (2).

$$F^* = \frac{\delta \mathcal{L}}{\delta H} = 0 \quad (26)$$

The Euler-Lagrange operator to the variable H , denoted by $\frac{\delta}{\delta H}$, is defined as follows:

$$\frac{\delta}{\delta H} = \frac{\partial}{\partial H} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha H)} - D_s \frac{\partial}{\partial H_s} - D_v \frac{\partial}{\partial H_v} + D_{ss} \frac{\partial}{\partial H_{ss}} + D_{vs} \frac{\partial}{\partial H_{vs}} + D_{vv} \frac{\partial}{\partial H_{vv}} \quad (27)$$

where $(D_t^\alpha)^*$ is the fractional derivative D_t^α adjoint operator. One way to express the RLD adjoint operator is as follows:

$$(oD_t^\alpha)^* = (-1)^n t I_T^{n-\alpha} (D_t^n) \equiv {}^C D_T^\alpha \quad (28)$$

The Caputo time-fractional derivative on the right side is defined as: ${}^C D_T^\alpha$.

$${}^C D_b^\alpha f(t) = (-1)^m t I_b^{m-\alpha} (D_t^m f(t)) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_t^b \frac{f^{(m)}(s)}{(s-t)^{\alpha-m+1}} ds \quad (29)$$

When the admissible $V(t, s, v, H)$ is not equal to zero and the adjoint equation (26) holds for all the solution of equation (2), after replacing in equation (2) some non-linearly self-adjoint equation. Substituting the formal Lagrangian (25) into this, yields the ensuing adjoint fractional (HJB) equation (Gazizov et al., 2015):

$$F^* = (D_t^\alpha)^* V - D_s(\mu s u) - D_v((rv + c)V) + D_{ss} \left(\frac{1}{2} k^2 s^{2\beta+2} V \right) + D_v \left(\frac{(\mu-r)^2 H_v}{k^2 s^{2\beta} H_{vv}} \right) + D_{vv} \left(\frac{(\mu-r)^2 H_v^2}{2k^2 s^{2\beta} H_{vv}^2} \right) - D_{vs} \left(\frac{k^2 s^{2\beta+2} H_{vs}}{H_{vv}} \right) D_{vv} \left(\frac{H_{vs}^2 k^2 s^{2\beta+2}}{H_{vv}^2} \right) + D_v \left(\frac{(\mu-r)s H_{vs}}{H_{vv}} \right) - D_{vs} \left(\frac{(\mu-r)s H_v}{H_{vv}} \right) + D_{vv} \left(\frac{(\mu-r)s H_v H_{vs}}{H_{vv}^2} \right) \quad (30)$$

After some simple computation, we can find all the injection of $V(t, s, v, H)$ for (HJB) equation non-linear self-adjointness and we get:

$$V(t, s, v, H) = \phi(t) \psi(t, s, v, H) \quad (31)$$

Where the function $\phi(t)$ is a consequence from

$$(oD_t^\alpha)^* \phi(t) = \left({}^C D_T^\alpha \phi(t) \right) = 0, \quad \phi(t) = C \quad (32)$$

C is an arbitrary constant. The substitution (t, s, v, H) is exactly the same as the classical (HJB) substitution: $\psi(t, s, v, H) = C$, where an arbitrary constant is denoted by C . Consequently, it has been demonstrated that the time-fractional (HJB) equation is non-linearly self-adjoint.

If part of fractional Noether's theorem (Lukashchuk, 2015), the components of conserved vector is obtained under a representation of Noether's operators acting over the Lagrangian. Given three independent variables: t, s and v which are the common independent coordinates with respect to H , one of the general forms of Noethers identity can be deduced in its simplest form.

$$\tilde{X} + D_t(\tau)\zeta + D_s(\xi)\zeta + D_v(\eta)\zeta = W \frac{\delta}{\delta H} + D_t N^t + D_s N^s + D_v N^v \quad (33)$$

I is the identity operator in the aforementioned identity, $\tilde{X}$ is a prolongation for the Lie symmetries (19), N^t, N^s and N^v are the Noether's operators, $\frac{\delta}{\delta H}$ stands for the Euler-Lagrange operator, and the characteristic of symmetry X is $W = \zeta - \tau H_t - \xi H_s - \eta H_v$. The following is the fractional Noether's operator N^t :

$$N^t = \tau \mathcal{L} + D_t^{\alpha-1}(W) \frac{\partial}{\partial(\partial_t^\alpha H)} + J \left(W, D_t \frac{\partial}{\partial(\partial_t^\alpha H)} \right) \quad (34)$$

$$J(h, \ell) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{h(\tau, x) \ell(\mu, x)}{(\mu-\tau)^{\alpha+1-n}} d\mu d\tau \quad (35)$$

$$D_t J(h, \ell) = h_t I_t^{n-\alpha} \ell - \ell \partial_t I_t^{n-\alpha} h \quad (36)$$

The Noether's operator sand v inidentity (33) are defined by the following relations respectively

$$N^s = \xi \mathcal{L} + W \left(\frac{\partial}{\partial H_s} - D_s \frac{\partial}{\partial H_{ss}} - D_v \frac{\partial}{\partial H_{vs}} \right) + D_v(W) \left(\frac{\partial}{\partial H_{vs}} \right) + D_s(W) \left(\frac{\partial}{\partial H_{ss}} \right) \quad (37)$$

$$N^v = \eta \mathcal{L} + W \left(\frac{\partial}{\partial H_v} - D_s \frac{\partial}{\partial H_{vs}} - D_v \frac{\partial}{\partial H_{ss}} \right) + D_s(W) \left(\frac{\partial}{\partial H_{vs}} \right) + D_v(W) \left(\frac{\partial}{\partial H_{ss}} \right) \quad (38)$$

If applying Noether's identity (33) to the formal Lagrangian (25) for arbitrary form of symmetry X of equation (2), and then any solution of this equation, from the equality it follows that right-hand side is: even if left-hand side equals zero:

$$D_t(N^t \mathcal{L}) + D_s(N^s \mathcal{L}) + D_v(N^v \mathcal{L}) \Big|_{(2)=0} = 0 \quad (39)$$

A conserved vector for any Lie symmetry of equation (2) may be found by comparing (39) and (24), with components specified by:

$$C^t = N^t \mathcal{L}, C^s = N^s \mathcal{L}, C^v = N^v \mathcal{L} \quad (40)$$

Thus, Eq. (40) may be used to generate the conserved vectors related to the Lie point symmetries of the TFHJBE.

Conservation laws for symmetry X_1 .

For the symmetry X_1 , the characteristics is as $W = -(c + rv)H_v$. Consequently, the symmetry X_1 yields the following conserved vectors using (40):

$$\begin{aligned} C^t &= c I_t^{1-\alpha} (-(c + rv)H_v) \\ C^s &= -(c + rv)H_v \left[\mu s C - \frac{1}{2} (2\beta + 2) k^2 s^{2\beta+1} C + \frac{C k^2 s^{2\beta+2}}{H_{vv}^2} (H_{vv} H_{vvs} - H_{vs} H_{vvv}) + \frac{(\mu-r)sC}{H_{vv}^2} (H_{vv}^2 - H_{vvv}) \right] + \\ &\quad (-rH_v - (c + rv)H_{vv}) \left(\frac{-C k^2 s^{2\beta+2} H_{vs}}{2H_{vv}} - \frac{Cs(\mu-r)H_v}{H_{vv}} \right) - (c + rv)H_{vs} \left(\frac{1}{2} k^2 s^{2\beta+2} C \right) \\ C^v &= -(c + rv)H_v \left[(c + rv) + \frac{(2\beta+2)C k^2 s^{2\beta+1} H_{vs}}{2H_{vv}} + \frac{C k^2 s^{2\beta+2} H_{vss}}{2H_{vv}} - \frac{C k^2 s^{2\beta+2} H_{vs} H_{vvs}}{2H_{vv}^2} - \frac{C(\mu-r)H_v}{H_{vv}} - \frac{C(\mu-r)sH_{vs}}{H_{vv}} + \right. \\ &\quad \left. \frac{C(\mu-r)sH_v H_{vvs}}{H_{vv}^2} \right] - (c + rv)H_{vs} \left(\frac{-C k^2 s^{2\beta+2} H_{vs}}{2H_{vv}} - \frac{Cs(\mu-r)H_v}{H_{vv}} \right) + (-rH_v - (c + rv)H_{vv}) \left(\frac{[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}]^2}{2k^2 s^{2\beta} H_{vv}^2} \right) \end{aligned}$$

Conservation laws for symmetry X_2 .

For this generator the characteristic is as $W = -sH_v$. So, using (40), conserved vectors C^t , C^s and C^v are given by

$$\begin{aligned} C^t &= c I_t^{1-\alpha} (-sH_v) \\ C^s &= (-sH_v) \left[\mu s C - \frac{1}{2} (2\beta + 2) k^2 s^{2\beta+1} C + \frac{C k^2 s^{2\beta+2}}{H_{vv}^2} (H_{vv} H_{vvs} - H_{vs} H_{vvv}) + \frac{(\mu-r)sC}{H_{vv}^2} (H_{vv}^2 - H_{vvv}) \right] + \\ &\quad + (-sH_{vv}) \left(\frac{-C k^2 s^{2\beta+2} H_{vs}}{2H_{vv}} - \frac{Cs(\mu-r)H_v}{H_{vv}} \right) + (-H_v - sH_{vs}) \left(\frac{1}{2} k^2 s^{2\beta+2} C \right) \\ C^v &= (-sH_v) \left[(c + rv) + \frac{(2\beta+2)C k^2 s^{2\beta+1} H_{vs}}{2H_{vv}} + \frac{C k^2 s^{2\beta+2} H_{vss}}{2H_{vv}} - \frac{C k^2 s^{2\beta+2} H_{vs} H_{vvs}}{2H_{vv}^2} - \frac{C(\mu-r)H_v}{H_{vv}} - \frac{C(\mu-r)sH_{vs}}{H_{vv}} + \frac{C(\mu-r)sH_v H_{vvs}}{H_{vv}^2} \right] - \\ &\quad (H_v + sH_{vs}) \left(\frac{-C k^2 s^{2\beta+2} H_{vs}}{2H_{vv}} - \frac{Cs(\mu-r)H_v}{H_{vv}} \right) + (-sH_{vv}) \left(\frac{[(\mu-r)H_v + k^2 s^{2\beta+1} H_{vs}]^2}{2k^2 s^{2\beta} H_{vv}^2} \right) \end{aligned}$$

Conservation laws for symmetry X_3

Finally for the generator X_3 the characteristic is as $W = -H$. Thus using (40), conserved vectors are as

$$C^t = c I_t^{1-\alpha} (-H)$$

$$\begin{aligned} C^s &= (-H) \left[\mu s C - \frac{1}{2} (2\beta + 2) k^2 s^{2\beta+1} C + \frac{C k^2 s^{2\beta+2}}{H_{vv}^2} (H_{vv} H_{vvs} - H_{vs} H_{vvv}) + \frac{(\mu-r)sC}{H_{vv}^2} (H_{vv}^2 - H_{vvv}) \right] + \\ &\quad (-H_v) \left(\frac{-C k^2 s^{2\beta+2} H_{vs}}{2H_{vv}} - \frac{Cs(\mu-r)H_v}{H_{vv}} \right) + (-H_s) \left(\frac{1}{2} k^2 s^{2\beta+2} C \right) \end{aligned}$$

$$C^v = (-H) \left[(c + rv) + \frac{(2\beta+2)Ck^2s^{2\beta+1}H_{vs}}{2H_{vv}} + \frac{Ck^2s^{2\beta+2}H_{vss}}{2H_{vv}} - \frac{Ck^2s^{2\beta+2}H_{vs}H_{vvs}}{2H_{vv}^2} - \frac{C(\mu-r)H_v}{H_{vv}} - \frac{C(\mu-r)sH_{vs}}{H_{vv}} + \frac{C(\mu-r)sH_vH_{vvs}}{H_{vv}^2} \right] +$$

$$(-H_s) \left(\frac{-Ck^2s^{2\beta+2}H_{vs}}{2H_{vv}} - \frac{Cs(\mu-r)H_v}{H_{vv}} \right) + (-H_v) \left(\frac{[(\mu-r)H_v + k^2s^{2\beta+1}H_{vs}]^2}{2k^2s^{2\beta}H_{vv}^2} \right)$$

6. Conclusions

This work's primary goal is to demonstrate, step-by-step, how the TFHJBE issue may be studied and solved using the Lie symmetry approach. The methods presented are then used to derive the conservation rules for FDEs utilizing RLD of order $\alpha \in (0, 1)$, which do not have a standard Lagrangian structure. All things considered, the Lie group analysis method works well for getting precise answers to a variety of FDEs. These findings are expected to spur more investigation, resulting in more sophisticated and effective techniques for resolving fractional integro-differential equations (FIDEs). Future research might examine the use of numerical techniques to answer Eq. (1) and other uses of Lie symmetry analysis in mathematical finance issues.

7. References

- Agrawal, O. P. (2002). Formulation of Euler–Lagrange equations for fractional variational problems. *Journal of Mathematical Analysis and Applications*, 272(1), 368–379.
- Alqhtani, M., Saad, K. M., Shah, R., & Hamanah, W. M. (2023). Discovering novel soliton solutions for (3+ 1)-modified fractional Zakharov–Kuznetsov equation in electrical engineering through an analytical approach. *Optical and Quantum Electronics*, 55(13), 1149.
- Bluman, G. W., & Anco, S. C. (2002). Symmetry and Integration Methods for Differential Equations. *Applied Mathematical Sciences*.
- Bluman, G. W., & Kumei, S. (2013). *Symmetries and Differential Equations*. Springer Science & Business Media.
- Butcher, J., Carminati, J., & Vu, K. (2003). A comparative study of some computer algebra packages which determine the Lie point symmetries of differential equations. *Computer physics communications*, 155(2), 92–114.
- Djordjevic, V. D., & Atanackovic, T. M. (2008). Similarity solutions to nonlinear heat conduction and Burgers/Korteweg–deVries fractional equations. *Journal of Computational and Applied Mathematics*, 222(2), 701–714.
- Fushchich, W. I., Shtelen, W., & Serov, N. I. (2013). *Symmetry analysis and exact solutions of equations of nonlinear mathematical physics* (Vol. 246). Springer Science & Business Media.
- Gazizov, R. K., Ibragimov, N. H., & Lukashchuk, S. Y. (2015). Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations. *Communications in Nonlinear Science and Numerical Simulation*, 23(1–3), 153–163.
- Gazizov, R. K., Kasatkin, A., & Lukashchuk, S. Y. (2007). Continuous transformation groups of fractional differential equations. *Vestnik Usatu*, 9(3), 21.
- Hashim, D. J., Jameel, A. F., & Ying, T. Y. (2023). Approximate solutions of fuzzy fractional differential equations via homotopy analysis method. *Progress in Fractional Differentiation and Applications*, 9(1), 167–187.
- Ibragimov, N. H. (2007). A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 333(1), 311–328.
- Ibragimov, N. K., & Kovalev, V. F. (2009). *Approximate and Renormgroup Symmetries*. Springer Science & Business Media.
- Jabber, A. M., & Eswad, M. H. N. (2025). Targeted Vaginal Prebiotic to Reduce Spontaneous Preterm Birth (sPTB) Biomarkers in Women with Prior sPTB: A Double-Blind RCT. *Letters in Biomathematics*, 12(1), 85–98.
- Jefferson, G., & Carminati, J. (2014). FracSym: automated symbolic computation of Lie symmetries of fractional differential equations. *Computer Physics Communications*, 185(1), 430–441.
- Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations* (Vol. 204). elsevier.
- Lukashchuk, S. Y. (2015). Conservation laws for time-fractional subdiffusion and diffusion-wave equations. *Nonlinear Dynamics*, 80(1), 791–802.

- Meleshko, S. V. (2006). *Methods for Constructing Exact Solutions of Partial Differential Equations: Mathematical and Analytical Techniques with Applications to Engineering*. Springer Science & Business Media.
- Podlubny, I. (1999). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*.
- Razminia, A., Asadizadehshiraz, M., & Torres, D. F. (2018). Fractional Order Version of the HJB Equation. *arXiv preprint arXiv:1810.12882*.
- Riewe, F. (1996). Nonconservative lagrangian and hamiltonian mechanics. *Physical Review E*, 53(2), 1890.
- Saad, K. M., & Srivastava, H. M. (2023). Numerical solutions of the multi-space fractional-order coupled Korteweg–De Vries equation with several different kernels. *Fractal and fractional*, 7(10), 716.
- Singla, K., & Gupta, R. K. (2017). Generalized Lie symmetry approach for fractional order systems of differential equations. III. *Journal of Mathematical Physics*, 58(6).
- Sun, X., Yong, X., & Gao, J. (2020). Optimal portfolio for a defined-contribution pension plan under a constant elasticity of variance model with exponential utility. *Frontiers of Mathematics in China*, 15(5), 1001-1009.
- Vu, K., Jefferson, G., & Carminati, J. (2012). Finding higher symmetries of differential equations using the MAPLE package DESOLVII. *Computer Physics Communications*, 183(4), 1044-1054.
- Wang, G.-w., Liu, X.-q., & Zhang, Y.-y. (2013). Lie symmetry analysis to the time fractional generalized fifth-order KdV equation. *Communications in Nonlinear Science and Numerical Simulation*, 18(9), 2321-2326.