

α -Cut Neutrosophic Z-Numbers with Shapley Improved Weighted Heronian Mean and its Algebraic Properties

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ABSTRACT

One of the most powerful aggregation factors is the Heronian Mean (HM) factor, which is based on studying the mutual relationships between the input elements and has been applied on a wide scale by decision makers in the case of multi-criteria decision-making (MCDM) problems in uncertain environments. Despite their usefulness, HM operators face some significant limitations in expressing complex relationships between input variables and are sometimes hampered by incomplete information management. To address such limitations, this study presents a new decision-making approach based on the Shapley fuzzy metric (SFM) and an improved generalized weighted HM operator (IGWHM) in the context of α -cut neutrosophic z-numbers (α -NZNs). This improved technique combines z-number theory, neutrosophic logic, and α -cut analysis. This approach promotes a more accurate and scalable means of representing uncertainty and reliability by defining α -cut neutrosophic z-numbers across different alpha levels. This reduces information loss during reasoning and enhances the capture of complex relationships between decision criteria. Therefore, this study introduces new operator: α -cut neutrosophic z-numbers Shapley improved generalized weighted Heronian mean (α -NZN-SIGWHM). This operator introduces the Shapley fuzzy value with Shapley mean in the α -NZN environment, providing an effective solution to MCDM problems based on inaccurate data. Furthermore, the influence of k and g on the clustering results is tested, observing their significant impact on decision-making outcomes. Finally, a numerical case study is presented and fully implemented to validate the effectiveness of the proposed method. The results confirm the consistency of the classifications obtained using different parameters. Overall, the results demonstrate that the proposed approach based on α -NZN performs effectively across decision criteria, providing a more comprehensive and reliable tool for solving complex real-world MCDM problems.

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1. Introduction

In the decision-making environment, decision makers (DMs) are constantly challenged when classifying alternatives by numerical values due to the inherent complexity and uncertainty of real-world problems. To address these problems, the concept of a fuzzy set was introduced in Zadeh (1965), which expresses fuzziness through a single degree of membership within the interval $[0, 1]$. Since then, numerous methods and mathematical expressions for fuzzy sets have been proposed, such as the intuitionistic fuzzy set (IFS), the interval-valued fuzzy set (Zadeh, 1975), and the Pythagorean fuzzy set (PFS) (Yager, 2013). Despite these advances, many traditional fuzzy set proposals still struggle to handle the unlimited and incomplete data typically encountered in real-world decision-making environments.

To address such problems, the neutrosophic set (NS) was proposed in Smarandache (1998) as a generalization of fuzzy and intuitionistic sets. NS is defined by three separate membership variables: truth, indeterminacy, and error, which helps model incomplete and inconsistent data to a greater degree. Due to its non-standard interval $[0, 1^+]$, the neutrosophic set is difficult to handle in computational decision-making mechanisms. To simplify the process, the single-valued neutrosophic set (SVNS) (Wang et al., 2010) and the interval neutrosophic set (INS) were introduced, where membership degrees are defined by upper and lower intervals to facilitate the representation of uncertainty.

Following the continuous changes, fuzzy set theory has been generally presented in the preferences of DMs. The fuzzy set (Dubois & Prade, 1978) was presented as a generalization of the fuzzy set; however, the ability to translate fuzzy sets is still effective. To overcome this limitation, the neutrosophic (NV) fuzzy set was studied in Alkhazaleh (2015) and was widely introduced for its mathematical properties. However, many problems still remain in the decision-making

process, as there are shortcomings in describing the degrees of truth, falsehood, and indeterminacy with single numerical values but with intervals. Therefore, the concept of α -cut neutrosophic z-numbers (α – NZNs) was introduced (Hashim et al., 2019), which helps in both interval-neutrosophic and fuzzy sets. The α – NZN technique uses interval-based membership functions that help reveal useful uncertainty details and reduce missing information during the classification process. By comparing with NV and INS, α – NZNs enhance the ability to translate data more widely and is particularly powerful in uncertain multi-criteria decision making (MCDM) (Al-Quran et al., 2020).

In addition, aggregation operators (AOs) are a mainstay of MCDM, combining multiple features into a single expressive value. Among these, the Heronian mean operator (HM) (Beliakov et al., 2007) is particularly powerful and robust in representing the interrelationships between input elements. Furthermore, several extensions have been introduced, such as the generalized Heronian mean (GHM) (Sýkora, 2009), the intuitionistic interval-valued fuzzy Heronian mean (Yu & Wu, 2012), and the hesitant double Pythagorean fuzzy Heronian mean (Tang et al., 2019). Further improvements include the linguistic fuzzy Heronian mean operators (Shi et al., 2019) and the weighted Heronian mean operators (Peng et al., 2018). Despite these advances, traditional Heronian mean operators can sometimes be poor at translating interactions between features, leading to easy information scatter.

To solve the problem of dispersion, the Shapley fuzzy measure (SFM) is applied to create a correlation between the overall interactions of decision attributes in the clustering process. This measure, introduced in cooperative game theory (SHAPLEY, 1953), has been widely used in decision-making processes to evaluate the weights of criteria (Lin et al., 2019; Wan et al., 2020). By taking into account the correlations with incomplete weight information, the use of the Shapley fuzzy measure enriches the interpretation of each criterion more accurately than classical weighting techniques (Hamza, 2025).

Moreover, although the HM and SFM metrics were introduced into SVNS in Tian et al. (2020), this concept still faces many gaps in effectively capturing uncertainty and incompleteness. To fill these gaps, this study introduces new aggregation operator, namely α -cut neutrosophic z- numbers Shapley improved generalized weighted Heronian mean (α – NZN – SIGWHM). This operator combines the fuzzy Shapley measure with the Heronian Shapley mean in the α -NZN environment, helping to create a more flexible and effective MCDM problem.

Finally, this paper is structured as follows. Section 2 introduces the definitions, operational laws, and degree functions of α -NZNs, the key concepts of fuzzy metrics, Shapley values, and Heronian mean operators. In Section 3, we expand the formulation of the proposed α – NZN – SIGWHM operator and explain their theoretical properties. Sections 4 and 5 apply the proposed operators to MCDMs. Section 6 presents an analysis of the effects of the g and k parameters and a comparative analysis. Finally, Section 7 concludes the study and outlines potential directions for future research.

2. Literature Review

Aggregation factors (AOs) play an effective role in decision-making processes and issues, enabling multiple evaluation criteria to be combined into a single template. Furthermore, several well-known aggregation factors have been proposed to aggregate diverse evaluation data with high accuracy, the most important of which are the arithmetic mean (AM) and the geometric mean (GM), which is primarily used in precise problems. The arithmetic mean (AM) focuses on collective opinion, while the geometric mean (GM) focuses on individual decisions (Pólya & Szegő, 1972).

Several researchers have developed aggregation in the decision-making process, adding their weighting proposals to these factors. The geometric mean was proposed by Saaty (1980) and the weighted average (WG) in the application of ratio-based decisions, while Hwang and Yoon (2012) introduced the weighted average (WA), known as the simple additive weighting (SAW) method, which takes into account the dispersed importance scores among criteria.

Beyond these terms, generalized forms have gained the attention of researchers, such as ordered weighted average (OWA) operators and ordered weighted geometric (OWG) operators. The OWA operator, initially defined by Csiszar (2021) describes a weight vector that represents a decision maker's path toward correctness or error, with data ranked from highest to lowest in importance. The OWG operator was then introduced by Chiclana et al. (2002), who proposed combining the terms WG and OWA. Due to their widespread use, the OWA and OWG operators have been developed for broader domains and deal with various non-precision problems (Torra, 1997). Since then, these operators have focused primarily on algebraic summation, which reduces their efficiency in more complex summation environments. To overcome these problems, the Dombey T -norm and T -conorm were introduced (Dombi, 1982), which further improve efficiency by including operational constraints within uncertain information. Since then, these operations have been studied on more operators in numerous clustering frameworks. Shi and Ye (2018) introduced Dombey operations by extending the clustering operators for neutrosophic information, introducing the Dombey weighted arithmetic mean and the Dombey weighted geometric mean operators. Kamaci et al. (2021) applied the Dombey T -norm and T -conorm to a neutrosophic dipole set to construct more accurate clustering operators.

Another influential area of clustering operators is the power average (PA), proposed by Yager (2001). The PA operator, Power Average (POWA), is a nonlinear weighting mechanism that allows data values to reinforce each other during the clustering process. This property makes PA operators particularly efficient in correlations between criteria. Liu and Tang

(2016) extended the PA operator by using an elasticity parameter (T), providing greater effectiveness in handling decision-making situations. In addition, Liu and Luo (2016) presented practical schemes of force aggregation operators in simplified neutrosophic group (SNS) problems and applied them to investment decision problems, demonstrating their practical effectiveness.

3. Preliminaries

In this section, the main definitions of the α – NZN – SIGWHM aggregation operator are briefly introduced.

3.1 α -NZN and its Operations

Definition 1. Let X be a reference set and x be generic element in X , the α – NZNs at a level α is defined as follows

$$\tilde{S} = [x, \langle (T_V^\alpha(x), T_R^\alpha(x)), (I_V^\alpha(x), I_R^\alpha(x)), (F_V^\alpha(x), F_R^\alpha(x)) \rangle, x \in X].$$

where $T_V^\alpha(x)$: Truth value at α -cut level, with $T_R^\alpha(x)$ as its associated reliability at α –level, $I_V^\alpha(x)$: Indeterminacy value at α -cut level, with $I_R^\alpha(x)$ as its associated reliability at α -level, $F_V^\alpha(x)$: Falsity value at α -cut level, with $F_R^\alpha(x)$ as its associated reliability at α –level. Each value component lies within $[0,1]$ for a given $\alpha \in [0,1]$, $0 \leq T_V^\alpha(x) + I_V^\alpha(x) + F_V^\alpha(x) \leq 3$ and $0 \leq T_R^\alpha(x) + I_R^\alpha(x) + F_R^\alpha(x) \leq 3$.

The operational laws of α -NZN are defined as follows.

Definition 2. Let $\tilde{A} = \langle (T_{V_1}^\alpha, T_{R_1}^\alpha), (I_{V_1}^\alpha, I_{R_1}^\alpha), (F_{V_1}^\alpha, F_{R_1}^\alpha) \rangle$, and $\tilde{B} = \langle (T_{V_2}^\alpha, T_{R_2}^\alpha), (I_{V_2}^\alpha, I_{R_2}^\alpha), (F_{V_2}^\alpha, F_{R_2}^\alpha) \rangle$, be two α -NZN Then,

i) Addition

$$\tilde{A} \oplus \tilde{B} = \langle (T_{V_1}^\alpha + T_{V_2}^\alpha - T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha + T_{R_2}^\alpha - T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha F_{R_2}^\alpha) \rangle. \quad (1)$$

ii) Multiplication

$$\tilde{A} \otimes \tilde{B} = \langle (T_{V_1}^\alpha T_{V_2}^\alpha, T_{R_1}^\alpha T_{R_2}^\alpha), (I_{V_1}^\alpha + I_{V_2}^\alpha - I_{V_1}^\alpha I_{V_2}^\alpha, I_{R_1}^\alpha + I_{R_2}^\alpha - I_{R_1}^\alpha I_{R_2}^\alpha), (F_{V_1}^\alpha + F_{V_2}^\alpha - F_{V_1}^\alpha F_{V_2}^\alpha, F_{R_1}^\alpha + F_{R_2}^\alpha - F_{R_1}^\alpha F_{R_2}^\alpha) \rangle. \quad (2)$$

iii) Scalar Multiplication

$$\lambda \tilde{A} = \langle (1 - (1 - T_V^\alpha)^\lambda, 1 - (1 - T_R^\alpha)^\lambda), (I_V^\alpha)^\lambda, (I_R^\alpha)^\lambda, (F_V^\alpha)^\lambda, (F_R^\alpha)^\lambda \rangle. \quad (3)$$

where $\lambda > 0$.

iv) Power

$$(\tilde{A})^\lambda = \langle (T_V^\alpha)^\lambda, (T_R^\alpha)^\lambda, (1 - (1 - I_V^\alpha)^\lambda, 1 - (1 - I_R^\alpha)^\lambda), (1 - (1 - F_V^\alpha)^\lambda, 1 - (1 - F_R^\alpha)^\lambda) \rangle. \quad (4)$$

where $\lambda > 0$

To compare the two presented α -NZN, the function score of α -NZN utilized in the study. It was conducted by Şahin (2014) which represented as the following.

Definition 3 (Al-Quran et al., 2020). Let $\tilde{A}$ in previous definition be α -NZN, then the score function $Y_{(\tilde{A})}$ is defined follows:

$$Y_{(\tilde{A}_i)} = \frac{\left(\frac{T_{V_i}^\alpha + T_{R_i}^\alpha}{2} \right) + 1 - \left(\frac{I_{V_i}^\alpha + I_{R_i}^\alpha}{2} \right) + 1 - \left(\frac{F_{V_i}^\alpha + F_{R_i}^\alpha}{2} \right)}{3}, \text{ for } Y_{(\tilde{A}_i)} \in [0, 1] \quad (5)$$

Respectively, if $Y_{(\tilde{A})} > Y_{(\tilde{B})}$, where $Y_{(\tilde{A})} \in [0, 1]$ and supposed $\tilde{A}$ and $\tilde{B}$ are two α -NZN then $\tilde{A}$ is bigger than $\tilde{B}$, represented by $\tilde{A} > \tilde{B}$.

3.2 Shapley Fuzzy Measure

Shapley measure that considers as a significant method in capturing the connection between the components was

firstly introduced by Sugeno (1974). This approach has been considered in different areas such as game theory and decision-making. The terms of λ -fuzzy measure are represented as the following:

Definition 4. (Sugeno, 1974) Let $P(N)$ be a set of function of $N, N = \{1, \dots, n\}$. Then a fuzzy measure on N is a set function $\mu: P(N) \rightarrow [0, 1]$ which satisfies:

- i. Boundary condition: $\mu(\emptyset) = 0$ and $\mu(N) = 1$.
- ii. Monotonicity: if $S, T \in P(N)$, then $\mu(S) \subseteq \mu(T)$.

Aiming to have simplicity, Sugeno (1974) introduced a significant type of fuzzy measure that is λ -fuzzy measure $P(N)$.

$$\mu(S \cup T) = \mu(S) + \mu(T) - \lambda \mu(S) \cdot \mu(T) \quad (6)$$

Where $S \cap T = \emptyset$ for all $S, T \in P(N)$ and $\lambda > -1$.

1) If $\lambda = 0$, then $\mu(S \cup T) = \mu(S) + \mu(T)$. This issue is known as an additive measure that does not indicate any interaction with S and T .

2) If $\lambda > 0$ then $\mu(S \cup T) > \mu(S) + \mu(T)$. This issue shows interaction positively synergic occurs with S and T and the set S, T have a multiplicative effect.

3) If $\lambda < 0$, then $\mu(S \cup T) < \mu(S) + \mu(T)$. The set S, T has a substitutive impact and it has a negatively synergic connection with S and T . Feather, fuzzy measure will be represented as below:

$$\mu_\lambda(S) = \begin{cases} \frac{1}{\lambda} (\prod_{C_i \in S} (1 + \lambda \mu_\lambda(\{C_i\})) - 1) & , \text{if } \lambda \neq 0 \\ \sum_{C_i \in S} \mu_\lambda(\{C_i\}) & , \text{if } \lambda = 0 \end{cases} \quad (7)$$

and the λ value will be obtained by finding the solution of the equation:

$$\prod_{C_i \in S} (1 + \lambda \mu_\lambda(\{C_i\})) = \lambda + 1 \quad (8)$$

According to the fuzzy measures and fuzzy integral, SHAPLEY (1953) proposed Shapley fuzzy measure. The λ -fuzzy measure has a significant impact on the measure of Shapley fuzzy. However, traditional weighting methods, the fuzzy measure can interpret each criterion individually. Based on the definition of the fuzzy measure shown in Definition 5, the fuzzy Shapley measure can be defined as

$$\varphi_i(\mu, N) = \sum_{\substack{j=1 \\ i \neq j}}^n \frac{(|N| - |V| - 1)! |V|!}{|N|!} (\mu(V \cup i) - \mu(V)), \forall i \in N \quad (9)$$

where μ possess a fuzzy measure on N . $|N|$ and $|V|$ represents the scalar of the set N and V . It is clear that $\varphi_i(\mu, N)$ expresses the weight vectors in the sett N as

$$\varphi_i(\mu, N) > 0 \text{ and } \sum_{\substack{j=1 \\ i \neq j}}^n \varphi_i(\mu, N) = 1 \quad (10)$$

3.3 The HERONIAN Mean (HM)

Definition 5 (Sýkora, 2009). Let the HM factor of dimension n be the HM of the assignment: $N^m \rightarrow N$ such that.

$$HM(m_i) = HM(m_1, \dots, m_n) = \frac{2}{n(n+1)} \sum_{\substack{i=1 \\ i \neq j}}^n \sqrt{m_i m_j} \quad (11)$$

where $N = [0, 1]$. Next, the formula HM denotes the Heronian mean (HM) operator.

Definition 6. (Sugeno, 1974) Let $g, k \geq 0$, and x_i ($i = 1, \dots, n$) be a set of positive numbers, $w_i = (w_1, \dots, w_n)$ is the weight vector of x_i satisfying $w_i > 0$ and $\sum_{\substack{j=1 \\ i \neq j}}^n w_i = 1$. If

$$GWHM^{g,k}(\check{S}_i) = \left(\frac{2}{n(n+1)} \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (w_i x_i)^g (w_j x_j)^k \right)^{\frac{1}{g+k}} \quad (12)$$

Then $GWHM^{g,k}$ denotes a generalized weighted Heronian mean (GWHM) operator.

Definition 7. (Sugeno, 1974) Let $g, k \geq 0$ and A_i be a set of positive numbers, $w_i = (w_1, \dots, w_n)$ is the weight vector of A_i where w_i shows the significant degree of A_i , satisfying $w_i > 0$ and $\sum_{\substack{j=1 \\ i \neq j}}^n w_i = 1$. Then $IGWHM^{g,k}$ is calculated as the following:

$$IGWHM^{g,k}(x_1, \dots, x_n) = \frac{(\sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n w_i w_j x_i^g x_j^k)^{\frac{1}{g+k}}}{(\sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n w_i w_j)^{\frac{1}{g+k}}} \quad (13)$$

Then $IGWHM^{g,k}$ denotes the improved generalized weighted Heronian mean (IGWHM) operator.

Definition 8. (Sugeno, 1974) Let $g, k \geq 0$, and A_i be a set of positive numbers. $w_i = (w_1, \dots, w_n)$ is the weight vector of A_i where w_i shows the significant degree of A_i , satisfying $w_i > 0$ and $\sum_{i \neq j}^n w_i = 1$. Then,

$$GGWHM(A_i) = \frac{1}{g+k} \prod_{i=1}^n \prod_{i \neq j}^n ((gx_i)^{w_i} + (kx_j)^{w_j})^{\frac{2}{n(n+1)}}, \quad (14)$$

Then $GGWHM^{g,k}$ denotes the generalized geometric weighted Heronian mean (IGWHM) operator.

Definition 9. (Sugeno, 1974). Let $g, k \geq 0$, and A_i be a set of positive numbers. $w_i = (w_1, \dots, w_n)$ is the weight vector of A_i where w_i shows the significant degree of A_i , satisfying $w_i > 0$ and $\sum_{i \neq j}^n w_i = 1$. Then,

$$IGWHM(A_i) = \frac{1}{g+k} \prod_{i=1}^n \prod_{i \neq j}^n (gx_i + kx_j)^{\frac{2(n+1-i)w_j}{n(n+1)\sum_{k \neq i}^n w_k}}, \quad (15)$$

Then $IGGWHM^{g,k}$ denotes the improved generalized geometric-weighted Heronian mean (IGWHM) operator.

4. Heronian Mean Operators with α -Cut Neutrosophic z-Numbers

This section presents HM and IGHM in the α -NZNs environment and develops two assembly operators, namely α -NZN-SIGWHM and α -NZN-SIGGWHM.

4.1 The α -NZN-SIGWHM Operator

Definition 10. Let $g, k \geq 0$, and A_i set of α -NZN. The Heronian mean factor for α -NZN s is expressed as

$$\alpha - NZN - SIGWHM = \left(\frac{\sum_{i=1}^n \left[\varphi_i(\mu, N) \varphi_j(\mu, N) (\check{S}_i^g \otimes \check{S}_j^k) \right]}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{g+k}} \quad (16)$$

where $\varphi_i(\mu, N)$ expresses the fuzzy Shapley measure and considers the fuzzy measure associated with μ on N satisfying $\varphi_i(\mu, N) \in [0, 1]$ and $\sum_{i \neq j}^n \varphi_i(\mu, N) = 1$, then $\alpha - NZN - SIGWHM^{g,k}$ denotes the α -cut neutrosophic z-numbers Shapley improved generalized weighted Heronian mean ($\alpha - NZN - SIGWHM$).

Theorem 1. Let $g, k \geq 0$ and $\check{S}_i = (T_{V_i}^\alpha, T_{R_i}^\alpha), (I_{V_i}^\alpha, I_{R_i}^\alpha), (F_{V_i}^\alpha, F_{R_i}^\alpha) (i = 1, \dots, n)$ set of α -NZN, resulting the aggregated value by the equation (17) additionally can be a α -NZN and

$$\begin{aligned} \alpha - NZN - SIGWHM^{g,k}(\check{S}_i) = & \left(\left(\left(\left(1 - \left(\prod_{j=1}^n, 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right), \left(1 - \right. \\ & \left. \left(\prod_{j=1}^n, 1 - (T_{R_i}^\alpha)^g (T_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \Bigg), \\ & \left(1 - \left(1 - \left(\left(\prod_{j=1}^n, 1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right), 1 - \left(1 - \left(\left(\prod_{j=1}^n, 1 - \right. \right. \right. \\ & \left. \left. \left(1 - I_{R_i}^\alpha \right)^g (1 - I_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \Bigg), \end{aligned}$$

$$\left(1 - \left(1 - \left(\left(\prod_{j=1}^n 1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}}, 1 - \left(1 - \left(\left(\prod_{j=1}^n 1 - (1 - F_{R_i}^\alpha)^g (1 - F_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right) \Bigg). \quad (17)$$

where $i, j = 1, \dots, n$.

Proof. By power law of $\alpha - NZN$ in Equation (7), we have

$$\begin{aligned} (\check{S}_i)^g &= \langle (T_{V_i}^\alpha)^g (T_{R_i}^\alpha)^g, (1 - (1 - I_{V_i}^\alpha)^g, 1 - (1 - I_{R_i}^\alpha)^g), (1 - (1 - F_{V_i}^\alpha)^g, \\ &\quad (1 - F_{R_i}^\alpha)^g) \rangle. \\ (\check{S}_j)^k &= \langle (T_{V_j}^\alpha)^k (T_{R_j}^\alpha)^k, (1 - (1 - I_{V_j}^\alpha)^k, 1 - (1 - I_{R_j}^\alpha)^k), (1 - (1 - F_{V_j}^\alpha)^k, \\ &\quad (1 - F_{R_j}^\alpha)^k) \rangle. \end{aligned}$$

Product rule of $\alpha - NZN$ in Equation (5), we have

$$\begin{aligned} (\check{S}_i)^g \otimes (\check{S}_j)^k &= \langle (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k, (T_{R_i}^\alpha)^g (T_{R_j}^\alpha)^k), (1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k, \\ &\quad 1 - (1 - I_{R_i}^\alpha)^g (1 - I_{R_j}^\alpha)^k), (1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k, 1 - (1 - F_{R_i}^\alpha)^g (1 - F_{R_j}^\alpha)^k) \rangle \end{aligned}$$

Consequently, using a scalar multiplication rule of α -NZN in Equation (6), the result is:

$$\begin{aligned} \varphi_i(\mu, N) \varphi_j(\mu, N) ((\check{S}_i)^g \otimes (\check{S}_j)^k) &= \left\langle \left(1 - \left(1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, 1 - \left(1 - (T_{R_i}^\alpha)^g (T_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right. \\ &\quad \left. , \left((1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, \right. \\ &\quad \left. (1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, \left((1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, \right. \\ &\quad \left. (1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right\rangle. \end{aligned}$$

To obtain equations when incorporating summation, by addition rule of $\alpha - NZN$ in Eq. (4), we have

$$\begin{aligned} \sum_{i \neq j}^n \varphi_i \varphi_j ((\check{S}_i)^g \otimes (\check{S}_j)^k) &= \left\langle \left(1 - \left(\prod_{j=1}^n 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, 1 - \left(\prod_{j=1}^n 1 - (T_{R_i}^\alpha)^g (T_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right. \\ &\quad \left. , \left(\prod_{j=1}^n (1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, \left(\prod_{j=1}^n (1 - (1 - I_{R_i}^\alpha)^g (1 - I_{R_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right. \\ &\quad \left. , \left(\prod_{j=1}^n (1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)}, \left(\prod_{j=1}^n (1 - (1 - F_{R_i}^\alpha)^g (1 - F_{R_j}^\alpha)^k)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right\rangle. \end{aligned}$$

Consequently, using the power rule of $\alpha - NZN$ in (4), the result is:

$$\begin{aligned}
& \frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)} \sum_{i \neq j} \varphi_i(\mu, N) \varphi_j(\mu, N) ((\check{S}_i)^g \otimes (\check{S}_j)^k) = \\
& \left\langle \left(\left(1 - \left(\prod_{i=1, j=1}^n 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right. \right. \\
& \left. \left(1 - \left(\prod_{i=1, j=1}^n 1 - (T_{R_i}^\alpha)^g (T_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right), \\
& \left(\left(\left(\prod_{i=1, j=1}^n \left(1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k \right) \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right. \\
& \left. \left(\left(\prod_{i=1, j=1}^n \left(1 - (1 - I_{R_i}^\alpha)^g (1 - I_{R_j}^\alpha)^k \right) \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right), \\
& \left(\left(\left(\prod_{i=1, j=1}^n \left(1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k \right) \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right. \\
& \left. \left(\left(\prod_{i=1, j=1}^n \left(1 - (1 - F_{R_i}^\alpha)^g (1 - F_{R_j}^\alpha)^k \right) \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right) \right\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_i) = \\
& \left\langle \left(\left(\left(1 - \left(\prod_{i=1, j=1}^n 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right. \right. \\
& \left. \left(1 - \left(\prod_{i=1, j=1}^n 1 - (T_{R_i}^\alpha)^g (T_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right), \\
& \left(1 - \left(1 - \left(\left(\prod_{i=1, j=1}^n \left(1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k \right) \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right. \\
& \left. \left(1 - \left(\left(\prod_{i=1, j=1}^n \left(1 - (1 - I_{R_i}^\alpha)^g (1 - I_{R_j}^\alpha)^k \right) \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \right) \right\rangle.
\end{aligned}$$

$$\left((1 - I_{R_i}^\alpha)^g (1 - I_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N) \left(\frac{1}{\sum_{j=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{g+k}}},$$

$$\left(1 - \left(1 - \left(\prod_{i=1, j=1}^n 1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N) \left(\frac{1}{\sum_{j=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{g+k}}} \right)^{\frac{1}{g+k}}, 1 \right.$$

$$\left. - \left(1 - \left(\prod_{i=1, j=1}^n 1 - (1 - F_{R_i}^\alpha)^g (1 - F_{R_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N) \left(\frac{1}{\sum_{j=1}^n \varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{g+k}}} \right)^{\frac{1}{g+k}} \right).$$

The Proposition of idempotency, monotonicity and boundedness of α -NZN-SIGWHM were introduced, accompanied with mathematical proofs. In what follows, we will prove that α -NZN-SIGWHM clustering operator has the following desirable properties.

The idempotency property is a crucial characteristic that maintains the operator's behavior when inputs are identical. If decision-makers provide the same truncated α -NZN, the aggregation process will produce the same final result $\check{S}_1 = \dots = \check{S}_i = \check{S}$, all the intermediate calculations such as Shapley weights and Heronian mean interactions become equal and constant, because there is no variation in the data, the operator has no reason to adjust or modify the values. Thus, the final aggregated result naturally remains.

$$\alpha\text{-NZN-SIGWHM}^{g,k}(\check{S}_i) = \check{S}$$

This means the operator does not distort uniform information, confirming that it behaves logically and reliably when all assessments agree.

Theorem 3 (Idempotency) Let $\check{S}_i (i = 1, \dots, n)$, set of α -NZN and $\check{S}_i = \check{S}$, for all i .

Then,

$$\alpha\text{-NZN-SIGWHM}^{g,k}(\check{S}_i) = \alpha\text{-NZN-SIGWHM}^{g,k}(\check{S}) = \check{S}. \quad (18)$$

Proof. Since $\check{S}_i = \check{S}$ for all i , we get

$$\begin{aligned} \alpha\text{-NZN-SIGWHM}^{g,k}(\check{S}_i) &= \left(\frac{\sum_{i,j=1}^m \varphi_i(\mu, N) \varphi_j(\mu, N)}{\sum_{i,j=1}^m \varphi_i(\mu, N) \varphi_j(\mu, N)} (\check{S}^g \otimes \check{S}^k) \right)^{\frac{1}{g+k}} = \left(\frac{\sum_{i,j=1}^m \varphi_i(\mu, N) \varphi_j(\mu, N)}{\sum_{i,j=1}^m \varphi_i(\mu, N) \varphi_j(\mu, N)} (\check{S}^{g+k}) \right)^{\frac{1}{g+k}} \\ &= \check{S} \left(\frac{\sum_{i,j=1}^m \varphi_i(\mu, N) \varphi_j(\mu, N)}{\sum_{i,j=1}^m \varphi_i(\mu, N) \varphi_j(\mu, N)} (\check{S}^{g+k}) \right)^{\frac{1}{g+k}} = \check{S}. \end{aligned}$$

Monotonicity is an important property that determines whether the coefficient α -NZN-SIGWHM is decreasing, increasing, or constant. A separate proof is then performed, and the proof is presented as shown.

Theorem 4 (Monotonicity). Let $\check{S}_i = \langle (T_{V_i}^\alpha, T_{R_i}^\alpha), (I_{V_i}^\alpha, I_{R_i}^\alpha), (F_{V_i}^\alpha, F_{R_i}^\alpha) \rangle$ and $\check{S}_i^* = \langle (T_{V_i}^{\alpha*}, T_{R_i}^{\alpha*}), (I_{V_i}^{\alpha*}, I_{R_i}^{\alpha*}), (F_{V_i}^{\alpha*}, F_{R_i}^{\alpha*}) \rangle$ be ant two sets of α -NZN and $(i = 1, \dots, n)$. If $\check{S}_i \leq \check{S}_i^*$, then

$$\alpha\text{-NZN-SIGWHM}^{g,k}(\check{S}_i) \leq \alpha\text{-NZN-SIGWHM}^{g,k}(\check{S}_i^*) \quad (19)$$

Proof.

1) For the truth part of α -NZN

When $T_{V_i}^\alpha \leq T_{V_i}^{\alpha*}$ and $T_{V_j}^\alpha \leq T_{V_j}^{\alpha*}$ at i and j with $g, k > 0$, we have

$$\begin{aligned}
 (T_{V_i}^\alpha)^g &\leq (T_{V_i}^{\alpha*})^g, (T_{V_j}^\alpha)^k \leq (T_{V_j}^{\alpha*})^k, \text{ then } 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \geq 1 - (T_{V_i}^{\alpha*})^g (T_{V_j}^{\alpha*})^k \\
 &\Rightarrow \left(\prod_{i \neq j}^n 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \geq \left(\prod_{i \neq j}^n 1 - (T_{V_i}^{\alpha*})^g (T_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \\
 &\Rightarrow 1 - \left(\prod_{i \neq j}^n 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \leq 1 - \left(\prod_{i \neq j}^n 1 - (T_{V_i}^{\alpha*})^g (T_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \\
 &\Rightarrow \left(1 - \left(\prod_{i \neq j}^n 1 - (T_{V_i}^\alpha)^g (T_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{g+k}} \leq \left(1 - \left(\prod_{i \neq j}^n 1 - (T_{V_i}^{\alpha*})^g (T_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{g+k}}.
 \end{aligned}$$

2) For the indeterminacy part of α -NZN

When $I_{V_i}^\alpha \leq I_{V_i}^{\alpha*}$ and $I_{V_j}^\alpha \leq I_{V_j}^{\alpha*}$ at i and j with $g, k > 0$, then we have

$$\begin{aligned}
 &\Rightarrow I_{V_i}^\alpha \leq I_{V_i}^{\alpha*}, I_{V_j}^\alpha \leq I_{V_j}^{\alpha*} \Rightarrow (I_{V_i}^\alpha)^g (I_{V_j}^\alpha)^k \leq (I_{V_i}^{\alpha*})^g (I_{V_j}^{\alpha*})^k \\
 &\Rightarrow 1 - (I_{V_i}^\alpha)^g (I_{V_j}^\alpha)^k \geq 1 - (I_{V_i}^{\alpha*})^g (I_{V_j}^{\alpha*})^k \\
 &\Rightarrow \left(\prod_{i \neq j}^n 1 - (I_{V_i}^\alpha)^g (I_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \leq \left(\prod_{i \neq j}^n 1 - (I_{V_i}^{\alpha*})^g (I_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \\
 &\Rightarrow 1 - \left(\left(\prod_{i \neq j}^n 1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \leq \\
 &1 - \left(\left(\prod_{i \neq j}^n 1 - (1 - I_{V_i}^{\alpha*})^g (1 - I_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \\
 &\Rightarrow \left(1 - \left(\left(\prod_{i \neq j}^n 1 - (1 - I_{V_i}^\alpha)^g (1 - I_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \leq \left(1 - \left(\left(\prod_{i \neq j}^n 1 - \right. \right. \right. \\
 &\left. \left. \left. (1 - I_{V_i}^{\alpha*})^g (1 - I_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}}.
 \end{aligned}$$

3) For the falsity part of α -NZN:

When $F_{V_i}^\alpha \leq F_{V_i}^{\alpha*}$ and $F_{V_j}^\alpha \leq F_{V_j}^{\alpha*}$ for all i and j with $g, k > 0$, then we have

$$\begin{aligned}
 &\left(1 - \left(\left(\prod_{i \neq j}^n 1 - (1 - F_{V_i}^\alpha)^g (1 - F_{V_j}^\alpha)^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}} \leq \left(1 - \left(\left(\prod_{i \neq j}^n 1 - \right. \right. \right. \\
 &\left. \left. \left. (1 - F_{V_i}^{\alpha*})^g (1 - F_{V_j}^{\alpha*})^k \right)^{\varphi_i(\mu, N) \varphi_j(\mu, N)} \right)^{\frac{1}{\sum_{i \neq j}^n \varphi_i(\mu, N) \varphi_j(\mu, N)}} \right)^{\frac{1}{g+k}}.
 \end{aligned}$$

The researchers also proposed the boundary hypothesis, which states that if α -NZN - SIGWHM is stable on a closed, bounded interval, then α -NZN - SIGWHM is a bounded function. Theorem 5 reveals this.

Theorem 5 (Boundedness). Let $\check{S}_i = \langle (T_{V_i}^\alpha, T_{R_i}^\alpha), (I_{V_i}^\alpha, I_{R_i}^\alpha), (F_{V_i}^\alpha, F_{R_i}^\alpha) \rangle, (i = 1, \dots, n)$ be a collection of α -NZN and $\check{S}_{\min} = \langle \min(T_{V_i}^\alpha, T_{R_i}^\alpha), \max(I_{V_i}^\alpha, I_{R_i}^\alpha), \max(F_{V_i}^\alpha, F_{R_i}^\alpha) \rangle$, $\check{S}_{\max} = \langle \max(T_{V_i}^\alpha, T_{R_i}^\alpha), \min(I_{V_i}^\alpha, I_{R_i}^\alpha), \min(F_{V_i}^\alpha, F_{R_i}^\alpha) \rangle$, then

$$\check{S}_{\min} \leq \alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_i) \leq \check{S}_{\max}$$

Proof. Since $\check{S}_i \geq \check{S}_{\min}$, we have

$$\alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_i) \geq \alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_{\min}, \check{S}_{\min}, \dots, \check{S}_{\min}) = \check{S}_{\min}.$$

Like, we can get

$$\alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_i) \leq \alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_{\max}, \check{S}_{\max}, \dots, \check{S}_{\max}) = \check{S}_{\max}$$

Then,

$$\check{S}_{\min} \leq \alpha\text{-NZN} - \text{SIGWHM}^{g,k}(\check{S}_i) \leq \check{S}_{\max} \quad (20)$$

5. MCDM Approach Based on α -NZN-SIGWHM Operator

An MCDM approach based on the proposed α -NZN-SIGWHM operator is developed in this section. Let $\check{A}_i = \{\check{A}_1, \check{A}_2, \dots, \check{A}_m\}$ and $C_j = \{c_1, c_2, \dots, c_n\}$ a set of alternatives and criteria. The collection of DMs is denoted by $H = \{h_1, h_2, \dots, h_\gamma\}$ an assessment is carried out by the pairwise comparison on i -th alternative with reflecting to j -th criterion using the linguistic variables of α -cut neutrosophic z-numbers. The successive phases of the presented decision-making method utilizing the α -NZN - SIGWHM are shown as the following:

Step 1. Create a decision matrix D , after surveying experts, $D = \langle T_{Z_{ij}}, I_{Z_{ij}}, F_{Z_{ij}} \rangle_{(m \times n)}$ in the MCDM.

Step 2. Set the value of α -cut to reduces the number of alternatives, then $D^\alpha = \langle T_{Z_{ij}}^\alpha, I_{Z_{ij}}^\alpha, F_{Z_{ij}}^\alpha \rangle_{(m \times n)}$.

Step 3. Shapley fuzzy measure.

The Shapley fuzzy measure can be obtained by Equations (7-8) and the fuzzy measure of each criterion is given by experts by Equations (9).

Step 4. Comprehensive evaluation.

To calculate the evaluation value of alternatives $A_i \{i = 1, \dots, m\}$, by the α -NZN-SIGWHM operator in Equation (17).

Step 5. Score function.

The score function for all alternatives $\check{A}_i$ is calculated using equation (5).

Step 6. Rank.

Order the alternatives using the derivative score function. The general proposed method in figure 1.

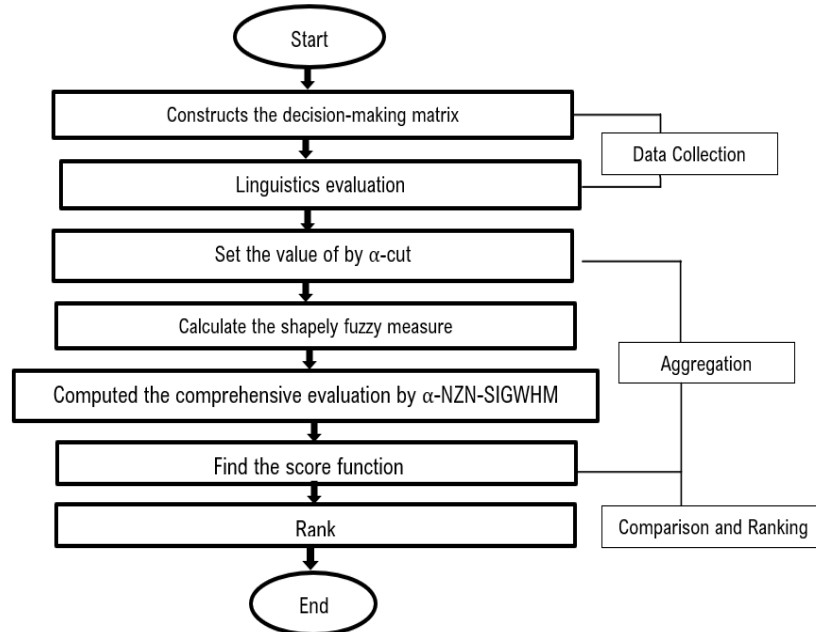


Figure 1: General Structure of the Proposed Method

6. Illustrative Example

This section illustrates an example of a company selecting a mobile device vendor to illustrate the applicability of the α -NZN – SIGWHM operator.

Suppose the company is tasked with choosing an appropriate supplier from several mobile device companies. The expert panel identifies four suppliers/alternatives, $A = \{A_1, A_2, A_3, A_4\}$, all offering mobile devices manufactured in 2023. There are three DMs $\{h_1, h_2, \dots, h_\gamma\}$. The selection process is based on three evaluation criteria: C_1 (the cost of device), C_2 (the quality of the device), and C_3 (the warranty of device). The four alternatives are found by experts according to the stated criteria and under α -cut neutrosophic z-numbers environment as represented in the decision matrix D .

6.1 The Procedure of the α -NZN-SIGWHM

Through the six steps and calculations that precede achieving the final arrangement of alternatives, we present them as follows:

Step 1. Create a decision matrix after surveying experts $D = \langle T_{z_{ij}}, I_{z_{ij}}, F_{z_{ij}} \rangle_{(m \times n)}$ in the MCDM.

$$D = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{matrix} A_i \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} \langle (T_{V_i}, T_{R_i}), (I_{V_i}, I_{R_i}), (F_{V_i}, F_{R_i}) \rangle & \langle (T_{V_i}, T_{R_i}), (I_{V_i}, I_{R_i}), (F_{V_i}, F_{R_i}) \rangle & \langle (T_{V_i}, T_{R_i}), (I_{V_i}, I_{R_i}), (F_{V_i}, F_{R_i}) \rangle \\ \langle (0.8, 0.8), (0.1, 0.8), (0.1, 0.7) \rangle & \langle (0.7, 0.7), (0.1, 0.7), (0.1, 0.9) \rangle & \langle (0.8, 0.6), (0.2, 0.7), (0.3, 0.7) \rangle \\ \langle (0.9, 0.7), (0.1, 0.4), (0.1, 0.2) \rangle & \langle (0.7, 0.7), (0.1, 0.1), (0.2, 0.3) \rangle & \langle (0.9, 0.8), (0.4, 0.2), (0.2, 0.4) \rangle \\ \langle (0.7, 0.6), (0.2, 0.1), (0.4, 0.2) \rangle & \langle (0.8, 0.8), (0.2, 0.3), (0.3, 0.1) \rangle & \langle (0.8, 0.7), (0.3, 0.4), (0.4, 0.3) \rangle \\ \langle (0.8, 0.8), (0.1, 0.3), (0.1, 0.4) \rangle & \langle (0.7, 0.7), (0.1, 0.3), (0.1, 0.2) \rangle & \langle (0.8, 0.6), (0.2, 0.3), (0.3, 0.2) \rangle \end{bmatrix} \end{matrix}$$

Step 2. Set the value of α -cut to reduces the number of alternatives, if $\alpha = 0.5$, then

$$D^{\alpha=0.5} = \langle T_{z_{ij}}^{\alpha}, I_{z_{ij}}^{\alpha}, F_{z_{ij}}^{\alpha} \rangle_{(m \times n)}$$

$$D^{\alpha=0.5} = \begin{matrix} & \begin{matrix} C_1 & C_2 & C_3 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} \langle (T_{V_i}^{\alpha}, T_{R_i}^{\alpha}), (I_{V_i}^{\alpha}, I_{R_i}^{\alpha}), (F_{V_i}^{\alpha}, F_{R_i}^{\alpha}) \rangle & \langle (T_{V_i}^{\alpha}, T_{R_i}^{\alpha}), (I_{V_i}^{\alpha}, I_{R_i}^{\alpha}), (F_{V_i}^{\alpha}, F_{R_i}^{\alpha}) \rangle & \langle (T_{V_i}^{\alpha}, T_{R_i}^{\alpha}), (I_{V_i}^{\alpha}, I_{R_i}^{\alpha}), (F_{V_i}^{\alpha}, F_{R_i}^{\alpha}) \rangle \\ \text{Excluded} & \text{Excluded} & \text{Excluded} \\ \langle (0.9, 0.7), (0.1, 0.4), (0.1, 0.2) \rangle & \langle (0.7, 0.7), (0.1, 0.1), (0.2, 0.3) \rangle & \langle (0.9, 0.8), (0.4, 0.2), (0.2, 0.4) \rangle \\ \langle (0.7, 0.6), (0.2, 0.1), (0.4, 0.2) \rangle & \langle (0.8, 0.8), (0.2, 0.3), (0.3, 0.1) \rangle & \langle (0.8, 0.7), (0.3, 0.4), (0.4, 0.3) \rangle \\ \langle (0.8, 0.8), (0.1, 0.3), (0.1, 0.4) \rangle & \langle (0.7, 0.7), (0.1, 0.3), (0.1, 0.2) \rangle & \langle (0.8, 0.6), (0.2, 0.3), (0.3, 0.2) \rangle \end{bmatrix} \end{matrix}$$

Step 3. The Shapley measure for all criteria is obtained through expert consensus as follows:

$$\mu(h_1) = 0.3, \mu(h_2) = 0.35, \mu(h_3) = 0.45$$

Since $\lambda \geq -1$ considered the only to be accepted, $\lambda = -0.26$ was chosen to be taken, based on Eq. (8), we get

$$\lambda + 1 = (1 + 0.4\lambda)(1 + 0.25\lambda)(1 + 0.45\lambda),$$

$$0.045\lambda^3 + 0.3925\lambda^2 + 0.1\lambda = 0 \Rightarrow \lambda \approx 0, -0.26, -8.15$$

By Eq. (8), the Shapley fuzzy measure of DMs was calculated as follows

$$\mu(h_1, h_2) = \frac{1}{\lambda}((1 + \lambda\mu(h_1))(1 + \lambda\mu(h_2)) - 1)$$

$$\mu(h_1, h_2) = \frac{1}{-0.26}((1 - 0.26(0.4))(1 - 0.26(0.25)) - 1)$$

$\mu(h_1, h_2) = 0.62$, by similar calculation, we have $\mu(h_1, h_3) = 0.71$, $\mu(h_2, h_3) = 0.76$, $\mu(h_1, h_2, h_3) = 1$.

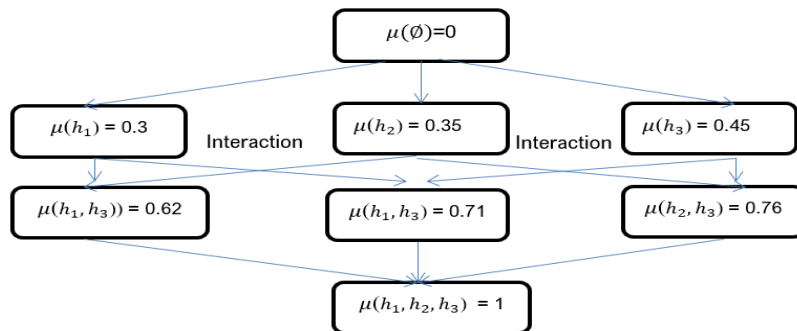


Figure 2: The Overall Fuzzy Measure

By applying Eq. (8), the fuzzy measure of the weight criteria, which obtained as follows

$$\mu(c_1) = 0.412, \mu(c_2) = 0.221, \mu(c_3) = 0.412.$$

With similar computation, the Shapley measure representing the weight of criteria is given as follows

$$\varphi_i(\mu, N) = \sum_{\substack{V \subseteq N/i \\ i \neq j}} \frac{(|N|-|V|-1)!|V|!}{|N|!} (\mu(V \cup i) - \mu(V)), \forall i \in N$$

$$\varphi_1(\mu, N) = \frac{(3-0-1)!0!}{3!} (\mu(h_1) - \mu(\emptyset)) + \frac{(3-1-1)!1!}{3!} ((\mu(h_1, h_2) - \mu(h_2)) + (\mu(h_1, h_3) - \mu(h_3))) + \frac{(3-2-1)!2!}{3!} (\mu(h_1, h_2, h_3) - \mu(h_2, h_3))$$

$$\varphi_1(\mu, N) = \frac{2!0!}{3!} (0.3 - 0) + \frac{1!1!}{3!} ((0.62 - 0.35)(0.71 - 0.45)) + \frac{0!2!}{3!} (1 - 0.76)$$

$$\varphi_1(\mu, N) = 0.27, \text{ in similar manner, we get } \varphi_2(\mu, N) = 0.32, \varphi_3(\mu, N) = 0.41.$$

In parallel, the Shapley measure, which represents the weight of the criteria, is calculated as follows

$$\varphi_1(\mu, N) = 0.4. \text{ In similar manner, we get } \varphi_2(\mu, N) = 0.2, \varphi_3(\mu, N) = 0.4.$$

Step 4. Calculate the total value of all alternatives using Eq. (17). When $g = k = 1$, we get

$$\check{A}_1 = \text{Excluded}.$$

$$\check{A}_2 = \langle (0.4032, 0.2492), (0.6683, 0.6953), (0.6121, 0.7986) \rangle.$$

$$\check{A}_3 = \langle (0.293, 0.2125), (0.717, 0.766), (0.7844, 0.6683) \rangle.$$

$$\check{A}_4 = \langle (0.2884, 0.2014), (0.547, 0.7914), (0.6131, 0.7438) \rangle.$$

Step 5. Eq. (8) is used to calculate the values of the score function.

$$\check{A}_1 \text{ is Excluded, } Y_{(S_{Z_2}^\alpha)} = 0.313, Y_{(S_{Z_3}^\alpha)} = 0.2616, Y_{(S_{Z_4}^\alpha)} = 0.2991.$$

Step 6. Rank the alternatives

The values of the score function form the basis for ranking the alternatives as $\check{A}_2 > \check{A}_4 > \check{A}_3$ and $\check{A}_1$ is excluded.

Therefore, the best alternative is $\check{A}_2$.

7. Effect of Parameters

To examine the effect of the parameters g and k on the results of alternatives under diverse values of g , k can be used. The consequence ranking output and the obtained score functions are represented in the following table:

Table 1: The Ordering of Alternatives for Varying g, k Values

g, k	$\alpha - NZN - SIGWHM$
$g = 1, k = 1$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 1, k = 2$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 2, k = 0$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 1, k = 3$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 5, k = 3$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 3, k = 5$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 10, k = 2$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 10, k = 1$	$\check{A}_2 > \check{A}_4 > \check{A}_3$
$g = 0.02, k = 1$	$\check{A}_2 > \check{A}_4 > \check{A}_3$

According to Table 1, the stability of the rankings across various values of g and k demonstrates that the decision results remain consistent and robust despite changes in parameter settings. This robustness is particularly important for practical MCDM applications dealing with uncertain or imprecise information, as the operators produce identical ranking orders under all tested parameter combinations. Such consistency indicates that the method effectively captures and aggregate the underlying information in a reliable manner, even though they differ in their structural and computational formulations. Therefore, the convergence of the ranking results suggests that incorporating the Shapley measure within the α -NZN framework enhances the efficiency and stability of the aggregation process across different parameter values.

8. Comparative Analysis

Table 2: (a) Comparison between Current Aggregation Operators

Method	Interaction of DMs and Criteria Weight	Aggregated Parameter	The Interrelationship between Input Arguments
SVNWAA	No	No	No
SVNWGA	No	No	No

Table 2: (b) Comparison between Current Aggregation Operators

Method	Interaction of DMs and Criteria Weight	Aggregated Parameter	The Interrelationship between Input Arguments
α -NZN-WA	No	No	No
α -NZN-WG	No	No	No
α -NZN-SNWBM	Yes	Yes	Yes
α -NZN-SNGBM	Yes	Yes	Yes
α -NZN-SIGWHM (Proposed Method)	Yes	Yes	Yes

Table 3: The Comparison with Existing Method

Method	Ranking
SVNWAA	$A_4 > A_2 > A_1 > A_3$
SVNWGA	$A_4 > A_1 > A_2 > A_3$
α -NZN-WA	$A_2 > A_4 > A_3$
α -NZN-WG	$A_4 > A_2 > A_3$
α -NZN-SNWBM ^{1,1}	$\tilde{A}_2 > \tilde{A}_4 > \tilde{A}_3$
α -NZN-SNGBM ¹	$\tilde{A}_4 > \tilde{A}_2 > \tilde{A}_3$
α -NZN-SIGWHM ^{1,1} (Proposed Method)	$\tilde{A}_2 > \tilde{A}_4 > \tilde{A}_3$

To demonstrate the advantages and reliability of the proposed method, several clustering methods in a truncated (α -NZN) environment were compared, including the proposed α -NZN – WA, α -NZN – WG, α -NZN – SIGWHM^{1,1}, α -NZN – SIGWHM^{1,1}, along with SVNWAA and SVNWGA. The results show that traditional methods such as SVNWAA and which rely on simple geometric and weighted averages, fail to account for inter-criteria relationships, often leading to biased classifications. Similarly, α -NZN-WG and α -NZN-SNGBM^{1,1} focus on multiplier effects, making them sensitive to extreme or inconsistent data. In contrast, the proposed methods integrate the Shapley Fuzzy Metric (SFM) with the Improved Generalized Heron Mean (IGHM), allowing them to observe synergistic and compensatory interactions between decision criteria. Consequently, these methods produce stable and consistent classifications ($\tilde{A}_2 > \tilde{A}_4 > \tilde{A}_3$) across different criteria, demonstrating greater robustness, reliability, and efficiency in handling uncertainties and MCDM problems compared to current methods.

9. Conclusion

In this paper, a new approach based on α – cut neutrosophic z – numbers (α – NZNs) is presented. This study aims to address decision-making problems based on uncertainty and interdependence in a MCDM environment. The study combines the Shapley measure (SFM) and the Heronian mean operator (HM) to efficiently link the relationships between evaluation criteria and decision-makers' preferences.

New aggregation operator was developed: α – cut neutrosophic z -numbers Shapley improved generalized weighted Heronian mean (α – NZN – SIGWHM). This operator leverages the advantages of α – cut-off levels, neutron representation, and fuzzy measurement theory to support the proposal and make it more comprehensive and flexible for decision-making.

The results derived from the proposed framework for a case study of mobile network selection decision-making reveal that proposed operator demonstrated consistent and effective results across various parameters, g and k . More specifically, the ranking of the alternatives remained unchanged, indicating the robustness and reliability of the model and their insensitivity to parameter changes. The identical ranking results for operator also demonstrate a high degree of similarity (100%), indicating that accurately.

By monitoring these high-level interactions between decisions criteria, the proposed α – NZN-based HM operators demonstrate an effective and accurate tool for decision-making in the face of uncertainty and complexity in data analysis and other practical applications. Future research could explore extending the operation of these operators to represent dynamic decision-making problems, integrating them with other models that provide deep optimization, or studying them for large-scale decision-making problems.

10. References

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