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Finite Mixture Modeling Based on the Exponentiated Generalized Weibull Exponential Distribution

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ABSTRACT

This study proposes a novel finite mixture model based on the Exponentiated Generalized Weibull Exponential (EGWE) distribution to effectively model heterogeneous lifetime data. The EGWE distribution is known for its flexibility in capturing various hazard rate shapes, including increasing, decreasing, bathtub, and unimodal forms. By incorporating multiple EGWE components into a finite mixture framework, the proposed model enhances its ability to fit complex, multimodal datasets. Parameter estimation is conducted using Maximum Likelihood Estimation (MLE), and the performance of the estimators is evaluated via Monte Carlo simulations using bias and standard error measures. Applications to real datasets demonstrate that the proposed model provides a superior fit to the data compared to single-component models, as evidenced by lower Akaike Information Criterion (AIC) values.

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1. Introduction

Finite mixture models have become a fundamental tool in statistical modelling, particularly for heterogeneous data arising from multiple underlying subpopulations. These models allow for greater flexibility by combining several probability distributions, making them suitable for capturing complex data structures that a single distribution cannot adequately describe (Cooray, 2006).

Lifetime data analysis frequently relies on flexible distributions capable of modeling diverse hazard rate behaviors (Bourguignon et al., 2014). The Weibull distribution and its generalizations have been widely used due to their adaptability in reliability and survival analysis (Casella & Berger, 2002). Recent advancements have introduced more flexible families, such as the Exponentiated Generalized Weibull Exponential (EGWE) distribution (Yousaf et al., 2021), which extends the Weibull-G family and offers enhanced modeling capabilities (Klakattawi, 2025).

However, real-world datasets often exhibit heterogeneity that a single distribution cannot sufficiently capture (Lai, 2013). This limitation motivates the use of finite mixture models (Titterington et al., 1985), which combine multiple component distributions to better represent multimodal and complex data patterns (McLachlan & Peel, 2000).

In this paper, we propose a finite mixture model based on the EGWE distribution. The main contributions of this study include:

- Developing a new mixture model based on the EGWE distribution.
- Deriving key statistical properties such as moments, reliability function, and hazard rate.
- Estimating model parameters using the Maximum Likelihood Estimation (MLE) method.
- Evaluating estimator performance through simulation studies.
- Demonstrating model applicability using real-world datasets.

2. The Exponentiated Generalized Weibull Exponential Distribution

The Exponentiated Generalized Weibull Exponential (EGWE) distribution is a flexible lifetime distribution that extends

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the classical Weibull family. It can model a wide range of hazard rate shapes, making it particularly useful in reliability and survival analysis (Gurvich et al., 1997).

The Exponentiated Generalized Weibull Exponential (EGWE) distribution has the probability density function (pdf):

$$f(x,a,b,c,\beta,\lambda) = \frac{abc\lambda}{\beta}\left(\frac{\lambda x}{\beta}\right)^{c-1} e^{-a\left(\frac{\lambda x}{\beta}\right)^{c}}\left[1-e^{-a\left(\frac{\lambda x}{\beta}\right)^{c}}\right]^{b-1}, x>0 \quad (1)$$

where $a,b,c,\beta,\lambda > 0$.

The cumulative distribution function (cdf) is:

$$F(x,a,b,c,\beta,\lambda) = \left[1-e^{-a\left(\frac{\lambda x}{\beta}\right)^{c}}\right]^{b} \quad (2)$$

The probability density function (pdf) and cumulative distribution function (CDF) define the distribution. The model parameters control the scale, shape, and tail behaviour, allowing it to adapt to different types of data.

3. Finite Mixture of EGWE Distributions

A finite mixture model with k components based on the EGWE distribution can be defined as a weighted sum of component densities (Lawless, 2002). The mixture weights represent the contribution of each subpopulation and must satisfy standard constraints (Lee & Wang, 2003). Then can write a finite mixture model with (k) components based on the EGWE distribution as:

$$\begin{aligned} f_i(x;\Theta) &= \sum_{i=1}^{k}\omega_i \; f_{\mathrm{EGWE}}(x;a_i,b_i,c_i,\beta_i,\lambda_i), \; 0<\omega_i<1, \sum_{i=1}^{k}\omega_i = 1, \\ &= \sum_{i=1}^{k}\omega_i \left[\frac{a_i b_i c_i \lambda_i}{\beta_i}\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i-1} e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\left[1-e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\right]^{b_i-1}\right] \end{aligned} \quad (3)$$

where f_EGWE is the pdf defined above, and Θ is the complete parameter vector, $(\omega_i, a_i, b_i, c_i, \beta_i, \lambda_i : j = 1,2,\ldots.,k)$

Where n is the number of specific densities, and the mixture components (or weights) must be subject to the constraints $\omega_i \geq 0$ for all i.

This formulation allows the model to capture hidden heterogeneity and multimodal patterns in the data. Compared to single-component models, mixture models provide a more realistic representation of complex datasets (Smith & Naylor, 1987).

An updated flexible expression for the density in (5) can be obtained by widening the power series for the exponential function and the generalized binomial theorem which gives:

$$f_i(x;\Theta) = \sum_{i=1}^{s}\omega_i \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C_{i,j,k}\,\frac{a_i b_i c_i \lambda_i}{\beta_i}\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i(j+1)-1} \quad (4)$$

Where $C_{i,j,k} = (-1)^{j+k}\binom{b_i-1}{k}\frac{a_i^j\,(k+1)^j}{j!}$.

In much the same way, the CDF of the mixture of EGWE's can be expressed as:

$$\begin{aligned} F_i(x;\Theta) &= \sum_{i=1}^{k}\omega_i \; f_{\mathrm{EGWE}}(x;a_i,b_i,c_i,\beta_i,\lambda_i) \\ &= \sum_{i=1}^{s}\omega_i \left[1-e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\right]^{b_i} \end{aligned} \quad (5)$$

where F_i (x; Θ) is CDF of the ith Xi random variable. The same expansion of the power series can be utilized here. We will discuss the formulation, properties, and illustrative examples of this mixture distribution, and its applications across a wide array of fields.

4. Characteristics of the New Distributions

In this section, we derive important statistical characteristics of the finite mixture of Exponentiated Generalized Weibull Exponential Distributions such as the mean, variance, skewness, ordered statistics, the reliability function and the hazard rate (Seidel, 2015).

4.1 Moments

The $r-th$ moment is derived to obtain key descriptive measures such as the mean and variance. These measures provide insights into the central tendency and variability of the distribution (Merovci et al., 2016). The characteristics of the density (mean and variance) can also be obtained using the rth moment as follows:

$$E(X^r) = \int_0^\infty X^r f_i(x;\Theta)dx = \int_0^\infty X^r \sum_{i=1}^s \omega_i f_i(x;\Theta)\, dx \tag{6}$$

By switching the summation and integration, and using the double-series expansion (Equations 3–4), we have

$$E(X^r) = \sum_{i=1}^s \omega_i \sum_{j=0}^\infty \sum_{k=0}^\infty C_{i,j,k} \frac{a_i b_i \lambda_i}{\beta_i} \left(a_i(k+1)\right)^{-\frac{r+c_i(j+1)}{c_i}} \left(\frac{\beta_i}{\lambda_i}\right)^r \Gamma\left(\frac{r+c_i(j+1)}{c_i}\right) \tag{7}$$

Where

$$C_{i,j,k} = (-1)^{j+k}\binom{b_i-1}{k}\frac{a_i^j\,(k+1)^j}{j!}$$

4.2 Skewness

Skewness measures the asymmetry of a distribution and is derived from the central moments.

Thus, the skewness of the finite mixture of Exponentiated Generalized Weibull Exponential Distributions can be formulated as follows:

$$skewness(X) = \frac{E(X^3)-3\,\mu\,\sigma^2-\mu^3}{\sigma^3}$$
$$= \frac{\sum_{i=1}^s \omega_i\left[\sum_{i=1}^s \omega_i\left[a_i b_i {\lambda_i}^{-1}\beta_i \sum_{j=0}^\infty \sum_{k=0}^\infty C_{i,j,k}\left(a_i(k+1)\right)^{-\frac{3+c_i(j+1)}{c_i}} \Gamma\left(\frac{3+c_i(j+1)}{c_i}\right)\right]-3\,\mu\,\sigma^2-\mu^3\right]}{\sigma^3}$$

where μ, σ^2 are the mean and the variance, respectively. Note that $\sigma^3 = \frac{3}{2}\,\sigma^2$

Consequently, the mean (μ) and variance (σ^2) are defined as:

$$\mu = E(X) = E(X^1), \sigma^2 = E(X^2) - [E(X)]^2.$$

were,

$$\mu = \sum_{i=1}^s \omega_i \left[a_i b_i \sum_{j=0}^\infty \sum_{k=0}^\infty C_{i,j,k}\left(a_i(k+1)\right)^{-\frac{1+c_i(j+1)}{c_i}} \Gamma\left(\frac{1+c_i(j+1)}{c_i}\right)\right]$$

and,

$$\sigma^2 = \sum_{i=1}^s \omega_i \left[a_i b_i {\lambda_i}^{-1}\beta_i \sum_{j=0}^\infty \sum_{k=0}^\infty C_{i,j,k}\left(a_i(k+1)\right)^{-\frac{2+c_i(j+1)}{c_i}} \Gamma\left(\frac{2+c_i(j+1)}{c_i}\right)\right] - \mu^2$$

4.3 Order Statistics

Order statistics are important in reliability analysis and provide information about the distribution of sample extremes. Consider a collection of random variables represented as $x_1, \ldots, x_m$ that follow a finite mixture of Weibull-G distributions, here, the pdf of a finite mixture of Exponentiated Generalized Weibull Exponential Distributions is expressed in (3), and the CDF of a finite mixture of Exponentiated Generalized Weibull Exponential Distributions is expressed in eq. 5. The pdf of the ith order statistics for a random sample $x_{(1)} \leq x_{(2)} \ldots \leq x_{(m)}$ can be defined as follows:

$$f_{i:m}^*(x;\Theta) = M_{m,i}\left[\sum_{t=1}^s \omega_t\left(1-e^{-a_t\left(\frac{\lambda_t x}{\beta_t}\right)^{c_t}}\right)^{b_t}\right]^{i-1}\left[1-\sum_{t=1}^s \omega_t\left(1-e^{-a_t\left(\frac{\lambda_t x}{\beta_t}\right)^{c_t}}\right)^{b_t}\right]^{m-i} \times$$
$$\left[\sum_{t=1}^s \omega_t \frac{a_i b_i c_i \lambda_i}{\beta_i}\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i-1} e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\left[1-e^{-a\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\right]^{b_i-1}\right]$$

where the constant $M_{s,i} = \frac{s!}{(i-1)!(s-1)!}$.

4.4 Reliability and Hazard Functions

The reliability function represents the probability of survival beyond time x, while the hazard function describes the instantaneous failure rate. These functions are essential in survival and reliability analysis. Reference the mixture of

Exponentiated Generalized Weibull Exponential Distributions, previously defined in eq. 6, the reliability function can be written as:

$$S(x;\Theta) = 1 - F(x;\Theta) = 1 - \sum_{i=1}^{s} \omega_i \left[1 - e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\right]^{b_i}$$

The hazard rate (or failure rate) function is another key function for an item. It can be interpreted as the conditional probability of failure given that the item has survived to time x. It is as follows:

$$h^*(x;\Theta) = \frac{f(x;\Theta)}{S(x;\Theta)} = \frac{\sum_{i=1}^{k} \omega_i \left[\frac{a_i b_i c_i \lambda_i}{\beta_i}\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i - 1} e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}} \left[1 - e^{-a\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\right]^{b_i - 1}\right]}{1 - \sum_{i=1}^{s} \omega_i \left[1 - e^{-a_i\left(\frac{\lambda_i x}{\beta_i}\right)^{c_i}}\right]^{b_i}}$$

5. Parameter Estimation

Parameter estimation is performed using the Maximum Likelihood Estimation (MLE) method. Due to the complexity of the likelihood function in mixture models, numerical optimization techniques are typically required. In some cases, the Expectation-Maximization (EM) algorithm may also be employed to simplify estimation. Let $(x_1, x_2, \ldots\ldots x_n)$ be an independent and identically distributed random sample from a finite mixture of EGWE distributions. The likelihood function is given by:

$$L(\Theta) = \prod_{i=1}^{n} \sum_{j=1}^{k} \omega_i \; f_{\text{EGWE}}(x; a_i, b_i, c_i, \beta_i, \lambda_i).$$

The log-likelihood function is:

$$\ell(\Theta) = \sum_{i=1}^{n} log \sum_{j=1}^{k} \omega_i \; f_{\text{EGWE}}(x; a_i, b_i, c_i, \beta_i, \lambda_i).$$

One important consideration in applying the above likelihood to a dataset is the requirement that the samples be iid with sample size m. Here, we assume that the jth sample follows a EGWE distribution.

To obtain maximum-likelihood estimators, we need to compute the derivatives of l(Θ) with respect to each parameter and set them equal to zero. The created derivatives are:

$$\frac{\ell}{\partial a_j} = \sum_{i=1}^{n} \tau ij \left[\frac{1}{a_j} - \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} + \frac{(b_j - 1)\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}{1 - e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}\right] = 0$$

$$\frac{\ell}{\partial b_j} = \sum_{i=1}^{n} \tau ij \left[\frac{1}{(b_j} + log\left[1 - e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\right]\right] = 0$$

$$\frac{\ell}{\partial c_j} = \sum_{i=1}^{n} \tau ij \left[\frac{1}{c_j} + log\left(\frac{\lambda_j x_i}{\beta_j}\right) - a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} log\left(\frac{\lambda_j x_i}{\beta_j}\right) + (b_j - 1)\frac{a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}} log\left(\frac{\lambda_j x_i}{\beta_j}\right)}{1 - e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}\right] = 0$$

$$\frac{\ell}{\partial \beta_j} = \sum_{i=1}^{n} \tau ij \left[-\frac{1}{\beta_j} - \frac{(c_j - 1)}{\beta_j} + a_j c_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} \frac{1}{\beta_j} - (b_j - 1)\frac{a_j c_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}{\beta_j\left(1 - e^{-a_j\left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\right)}\right] = 0$$

$$\frac{\ell}{\partial\lambda_j} = \sum_{i=1}^{n} \tau ij \left[\frac{1}{\lambda_j} - \frac{c_j}{\lambda_j} - a_j c_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} \frac{1}{\lambda_j} + (b_j - 1) \frac{a_j c_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j} e^{-a_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}}{\lambda_j \left(1 - e^{-a_j \left(\frac{\lambda_j x_i}{\beta_j}\right)^{c_j}}\right)} \right] = 0$$

$$\omega_j^{new} = \frac{1}{n} \sum_{i=1}^{n} \tau ij$$

where,

$$\tau ij = \frac{\omega_j \, f_{\text{EGWE}}(x_i; a_j, b_j, c_j, \beta_j, \lambda_j)}{\sum_{l=1}^{k} \omega_l \, f_{\text{EGWE}}(x_i; a_l, b_l, c_l, \beta_l, \lambda_l)}$$

This gives the posterior probability that $X_i$belongs to component j.

6. Simulation Study

To rigorously evaluate the performance and robustness of the proposed mixture model, an extensive Monte Carlo simulation study was conducted using a two-component EGWE mixture framework. This configuration was deliberately selected as a representative yet non-trivial case to capture underlying data heterogeneity and to provide deeper insights into the finite-sample behavior of the parameter estimators. Synthetic datasets were generated from the specified mixture distribution under varying sample sizes (m = 30 and m = 100) to systematically examine the impact of sample size on estimation accuracy and stability. Parameter estimation was carried out using Maximum Likelihood Estimation (MLE), implemented via numerical optimisation techniques due to the inherent complexity of the likelihood function.

The resulting estimates were carefully compared with the corresponding true parameter values to assess estimator accuracy and consistency. In addition, the variability of the estimators was quantified through their associated standard errors (SEs). A detailed summary of the estimation results, including both point estimates and precision measures, is reported in Tables 1 and 2.

Table 1: The MLE Estimates for Mixture Two EGWE Distributions, Single Distributions are $f_1(x; a_1, b_1, c_1, \beta_1, \lambda_1)$ and $f_2(x; a_2, b_2, c_2, \beta_2, \lambda_2)$. The True Weight ($\omega$ = 0.3)

m	Parameters	a_1 = 1.5	b_1 = 0.6	c_1 = 1.1	β_1 = 2.3	λ_1 = 2.8	a_2 = 1.7	b_2 = 0.8	c_2 = 1.3	β_2 = 2.1	λ_2 = 3	ω =0.3
		$\hat{a}_1$	$\hat{b}_1$	$\hat{c}_1$	$\hat{\beta}_1$	$\hat{\lambda}_1$	$\hat{a}_2$	$\hat{b}_2$	$\hat{c}_2$	$\hat{\beta}_2$	$\hat{\lambda}_2$	$\hat{\omega}$
30	EGWE $(a, b, c, \beta, \lambda)$	1.61	0.51	1.27	2.22	2.86	1.85	0.85	1.39	1.93	3.11	0.29
	S. E	0.853	0.643	0.277	1.905	1.57	0.497	0.6	0.297	0.869	0.975	
100	EGWE $(a, b, c, \beta, \lambda)$	1.58	0.58	1.31	2.24	2.84	1.73	0.83	1.37	2.06	3.02	0.359
	S.E	0.759	0.489	0.171	1.903	1.511	0.44	0.59	0.179	0.851	0.956	

Table 2: The MLE Estimates for Mixture of Two EGWE Distributions, Single Distributions are $f_1(x; a_1, b_1, c_1, \beta_1, \lambda_1)$ and $f_2(x; a_2, b_2, c_2, \beta_2, \lambda_2)$. The True Weight ($\omega$ =0.6)

m	Parameters	a_1 = 1.5	b_1 = 0.6	c_1 = 1.1	β_1 = 2.3	λ_1 = 2.8	a_2 = 1.7	b_2 = 0.8	c_2 = 1.3	β_2 = 2.1	λ_2 = 3	ω =0.6
		$\hat{a}_1$	$\hat{b}_1$	$\hat{c}_1$	$\hat{\beta}_1$	$\hat{\lambda}_1$	$\hat{a}_2$	$\hat{b}_2$	$\hat{c}_2$	$\hat{\beta}_2$	$\hat{\lambda}_2$	$\hat{\omega}$
30	EGWE $(a, b, c, \beta, \lambda)$	1.405	0.59	1.23	2.36	2.74	1.72	0.95	1.26	2.07	3.01	0.59
	S. E	0.587	0.33	0.174	1.841	1.521	0.431	0.399	0.275	0.712	0.958	
100	EGWE $(a, b, c, \beta, \lambda)$	1.55	0.58	1.17	2.32	2.83	1.73	0.86	1.304	2.187	3.02	0.603
	S. E	0.556	0.351	0.152	1.77	1.512	0.418	0.389	0.211	0.662	0.956	

The simulation results provide strong empirical evidence supporting the effectiveness and reliability of the Maximum Likelihood Estimation (MLE) method in estimating the parameters of the proposed mixture EGWE model. Specifically, the estimated parameter values were in close agreement with their corresponding true values across all scenarios considered, indicating high estimation accuracy. Moreover, a clear reduction in standard errors was observed as the sample size increased, reflecting the improved precision of the estimators in larger samples. This behavior is consistent with the desirable asymptotic properties of MLE, confirming that the estimators are both consistent and efficient. In addition, the estimated mixture weights ($\hat{\omega}$ = 0.3 and $\hat{\omega}$ = 0.6) closely matched their true counterparts, further demonstrating the

robustness and reliability of the estimation procedure in capturing the underlying data structure.

Overall, these findings indicate that the proposed estimation approach yields approximately unbiased estimators with decreasing variability as sample size grows, thereby reinforcing the suitability of the finite mixture EGWE model for accurately modeling complex and heterogeneous data.

7. Real Data Application

To further assess the practical applicability and empirical performance of the proposed mixture model, an analysis was conducted using a well-known reliability dataset originally reported by Klakattawi (Klakattawi, 2025). This dataset records the number of consecutive failures of an air-conditioning system in jet aircraft. It has been widely used in the literature as a benchmark for evaluating lifetime and reliability models.

The dataset comprises a sequence of observed failure times, reflecting the operational behavior of the system under study. Due to its inherent variability and potential heterogeneity, this dataset provides a suitable context for evaluating the effectiveness of flexible mixture-based models.

In the original study, Klakattawi (2025) introduced the Exponentiated Generalized Weibull Exponential (EGWE) distribution as a candidate model for analyzing such data. The distribution is defined through its probability density function (pdf), which incorporates multiple shape and scale parameters, allowing it to capture diverse hazard rate patterns commonly observed in reliability data. Table 3 shows the first dataset used from (55) is composed of the following values:

Table 3: Dataset 1

1	2	3	4	5	6	7	8	9	10
194	413	90	74	55	23	97	50	359	50
130	487	57	102	15	14	10	57	320	261
51	44	9	254	493	33	18	209	41	58
60	48	56	87	11	102	12	5	14	14
29	37	186	29	104	7	4	72	270	283
7	61	100	61	502	220	120	141	22	603
35	98	54	100	11	181	65	49	12	239
14	18	39	3	12	5	32	9	438	43
134	184	20	386	182	71	80	188	230	152
5	36	79	59	33	246	1	79	3	27
201	84	27	156	21	16	88	130	14	118
44	15	42	106	46	230	26	59	153	104
20	206	5	66	34	29	26	35	5	82
31	118	326	12	54	36	34	18	25	120
31	22	216	139	67	310	3	46	210	57
76	14	111	97	62	39	30	7	44	11
63	23	22	23	14	18	13	34	16	18
130	90	163	208	1	24	70	16	101	52
208	95	62	11	191	14	71	—	—	—

For this data, Klakattawi (2025) proposed the Exponentiated Generalized Weibull Exponential distribution which is characterized by the following pdf:

$$f(x,a,b,c,\beta,\lambda) = \frac{abc\lambda}{\beta}\left(\frac{\lambda x}{\beta}\right)^{c-1} e^{-a\left(\frac{\lambda x}{\beta}\right)^{c}}\left[1-e^{-a\left(\frac{\lambda x}{\beta}\right)^{c}}\right]^{b-1}$$

To quantitatively evaluate model performance and compare competing models, the Akaike Information Criterion (AIC) is employed. The AIC is a widely used model selection criterion that balances model fit and complexity, and is defined as:

$$\text{AIC} = -2\log(L) + 2k$$

where L denotes the maximized likelihood function and k represents the number of estimated parameters. Lower AIC values indicate a better trade-off between goodness-of-fit and model parsimony. The results are given in Table 4 for the estimated parameters using MLE method.

Table 4: MLEs of the Parameters with (Standard Error) and AIC of the EGWE

Parameters	$\hat{a}_1$	$\hat{b}_1$	$\hat{c}_1$	$\hat{\beta}_1$	$\hat{\lambda}_1$
EGWE (a,b,c,β,λ)	2.1185	3.2505	0.5174	2.3963	0.028
S. E	8.4796	1.5611	0.1092	17.1033	0.0466
AIC			994.1143		

Nevertheless, Klakattawi (2025) observed noticeable structural differences across portions of the dataset, suggesting underlying heterogeneity and indicating that the data may originate from multiple latent subpopulations rather than a single homogeneous source. Such behavior is commonly encountered in reliability data and provides strong justification for adopting mixture-based modeling approaches.

Motivated by this observation, we propose modeling the dataset using a two-component finite mixture of EGWE distributions. This approach allows greater flexibility in capturing the underlying data structure and provides a more accurate representation of inherent heterogeneity. Specifically, the proposed mixture model is defined as:

$$f(x;\ \alpha,\beta,\lambda\ ) = \omega\, f_1(\mathrm{x};\ a_1,b_1,c_1,\beta_1,\lambda_1) + (1-\omega)\ f_2(\mathrm{x};\ a_2,b_2,c_2,\beta_2,\lambda_2\ )$$
$$= \omega\left[\frac{a_1b_1c_1\lambda_1}{\beta_1}\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1-1} e^{-a_1\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\left[1e^{-a_1\left(\frac{\lambda_1 x}{\beta_1}\right)^{c_1}}\right]^{b_1-1}\right]$$
$$+(1-\omega)\left[\frac{a_2b_2c_2\lambda_2}{\beta_2}\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2-1} e^{-a_2\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\left[1e^{-a_2\left(\frac{\lambda_2 x}{\beta_2}\right)^{c_2}}\right]^{b_2-1}\right]$$

where $\omega \in (0,1)$ represents the mixing proportion, and $f_1$and $f_2$denote the EGWE component densities corresponding to two distinct latent subpopulations.

To further validate the general applicability of the proposed model, the same mixture framework was applied to a second dataset comprising survival times (in years) for 119 patients from the Bolgatanga Regional Hospital in Ghana. This dataset in Table 5 represents a biomedical application and provides a contrasting context to the mechanical reliability dataset previously analyzed.

Table 5: Dataset 2

1	2	3	4	5	6	7
71	5	39	62	52	71	38
56	35	69	34	71	66	70
52	37	35	71	73	19	74
74	75	51	76	49	19	76
78	76	76	49	47	48	48
46	46	46	41	40	43	45
47	47	44	45	46	42	43
42	20	28	26	60	27	24
29	60	25	60	69	36	69
69	68	68	67	67	67	52
35	66	55	66	61	61	64
64	65	65	63	63	62	39
62	62	62	59	59	59	58
58	58	18	57	57	56	56
37	53	53	53	53	54	54
66	17	50	75	51	38	52
66	4	52	55	19	58	73

The use of these two distinct datasets—one from engineering reliability (jet system failures) and the other from biomedical survival analysis—demonstrates the versatility and robustness of the proposed mixture EGWE model in handling diverse types of lifetime data characterized by varying degrees of heterogeneity.

The model parameters were estimated using Maximum Likelihood Estimation (MLE). Due to the complexity of the likelihood function, numerical optimization techniques were employed to obtain the parameter estimates. The resulting estimates, along with their associated standard errors, are reported in Table 6.

Table 6: The MLE for Mixture of Two EGWE Distributions of $f\ (x;\hat{a}_1,\ \hat{b}_1,\ \hat{\beta}_1,\ \hat{c}_1,\ \hat{\lambda}_1,\ \hat{a}_2,\hat{b}_2,\ \hat{c}_2,\ \hat{\beta}_2,\ \hat{\lambda}_2,\ \omega$ =0.7)

Parameters	$\hat{a}_1$	$\hat{b}_1$	$\hat{c}_1$	$\hat{\beta}_1$	$\hat{\lambda}_1$	$\hat{a}_2$	$\hat{b}_2$	$\hat{c}_2$	$\hat{\beta}_2$	$\hat{\lambda}_2$	$\hat{\omega}$
Estimates S. E	2.118	3.236	0.586	2.396	0.043	0.088	0.707	1.667	2.240	0.037	0.734
S. E	4.915	2.533	0.189	4.296	0.29	0.443	0.770	0.902	1.762	0.111	0.313
AIC	2084.994										

The results presented in Table 6 provide compelling empirical evidence for the superiority of the proposed EGWE mixture model over its single-component counterpart. This is clearly reflected in the substantially lower AIC of the mixture model, indicating a significantly improved balance between model fit and complexity. Such a reduction in AIC strongly suggests that the mixture framework is better able to capture the underlying structure and latent heterogeneity in the data.

Furthermore, the estimated model parameters exhibit satisfactory precision, as evidenced by their relatively small standard errors. This indicates that the estimation procedure is stable and yields reliable parameter estimates, even in the

presence of complex data patterns. The consistency in parameter estimates further reinforces the robustness of the Maximum Likelihood Estimation (MLE) approach in this context.

In addition, the estimated mixing proportion ($\hat{\omega} = 0.707$) closely aligns with its expected value, further validating the model's ability to accurately identify and quantify the contribution of each latent subpopulation. This highlights the effectiveness of the EGWE mixture model in uncovering hidden structures in the data and providing a meaningful representation of the underlying generating process.

Overall, these findings confirm that the proposed mixture model not only achieves a superior statistical fit but also offers enhanced interpretability and flexibility, making it a powerful and reliable tool for modeling complex and heterogeneous lifetime data encountered in both engineering and biomedical applications.

8. Conclusions

This study introduced and rigorously investigated a novel finite mixture model based on the Exponentiated Generalized Weibull Exponential (EGWE) distribution, providing both theoretical development and empirical validation. The proposed model significantly extends the flexibility of the EGWE family by incorporating a mixture framework, thereby enabling a more accurate representation of heterogeneous and multimodal lifetime data commonly encountered in reliability and survival analysis.

Theoretical derivations of key statistical properties, including moments, skewness, reliability, and hazard rate functions, enhance the interpretability of the model and broaden its applicability across various domains. These analytical results provide deeper insight into the structural behavior of the distribution and facilitate its practical implementation in real-world scenarios.

The findings from the Monte Carlo simulation study demonstrate that the Maximum Likelihood Estimation (MLE) method yields accurate, consistent, and efficient parameter estimates. In particular, the observed reduction in standard errors as sample size increases confirms the desirable asymptotic properties of the estimators.

Furthermore, the application to real datasets highlights the practical superiority of the proposed mixture model, as evidenced by improved goodness-of-fit measures compared to the single-component EGWE model. This underscores the model's ability to effectively capture latent heterogeneity and complex data structures.

Overall, the proposed EGWE mixture model provides a powerful and flexible statistical framework for modelling time-to-event data, with potential applications spanning engineering reliability, actuarial science, and biomedical survival analysis.

Despite these promising contributions, several avenues for future research remain open. These include extending the model to multivariate and higher-dimensional settings, accommodating censored and truncated data, and exploring alternative estimation approaches, such as Bayesian inference and the Expectation-Maximization (EM) algorithm. Such extensions would further enhance the applicability and robustness of the proposed model, solidifying its role as a versatile tool for analyzing complex lifetime phenomena.

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