

# Dynamic Stability of the Exponentiated Exponential Autoregressive Model with Application

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## ABSTRACT

This study conducts elaborate development and critique of the Exponentiated-Exponential Autoregressive Model (EEAR(p)). The formal model Equation has been defined theoretically, and the requirements for a unique nonzero fixed point are well established. In addition, a thorough stability condition was also specified. This criterion is based on analysing the roots of the characteristic Equation, the nonlinear transformation coefficients, and the h functions; in particular, the h functions require that all roots lie strictly within the unit circle. The analysis of the theoretical assumptions further confirmed that the EEAR model exhibits constant dynamics unless specific parameters for the combination of the linear and weighted exponential components are met. The EEAR model was empirically applied to weekly silver prices, with estimates up to the tenth order and evaluated using the information criteria (AIC, BIC, HQC, CAIC), along with a dynamic stability check via polynomial root analysis. Results showed that EEAR (5) was the most suitable specification, as it had the smallest AIC among stable models and the lowest residual variability. In addition, path diagrams show the practical application of EEAR stability (5), since the trajectory converges to a single consistent point, thereby significantly supporting predictability. These findings, combined with the findings presented, confirm that the EEAR (5) model has shown to be more flexible and better able to model the financial time series of interest, validating the effectiveness of a dynamic analysis in combination with statistical properties, to provide strong models to nonlinearity model formulation.

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## 1. Introduction

Nonlinear dynamic models are among the most widely used frameworks for examining complex phenomena over time, as they incorporate temporal dynamics, as well as stochastic and deterministic system effects. If financial or economic data are nonlinear, conventional linear models are impractical; thus, flexible mathematical formulations are required. Among them, the Exponentiated Exponential Autoregressive (EEAR) framework is one of the most widely adopted methodologies, given that the exponentiated exponential distribution is based on shape and scale parameters, which provide the system with greater ability to capture the real behaviour of time series with extreme variations or highly volatile effects. Ozaki (1980), the same researcher, established the dynamic stability of kinetic systems in 1980 and developed further in 1985 (Ozaki, 1985), followed by many studies which followed the same approach including: Stability of Some Nonlinear Autoregressive Models (Ahmad et al., 2025), Logistic Autoregressive Model (Weis, 1986), Cauchy Autoregressive Model (Kirch & Kamgaing, 2012), Burr X Autoregressive Model (Alzghool, 2017), exponential (GARCH) models (Youns & Ahmad, 2023), GJR-GARCH (Q, P) models (Noori & Mohammad, 2021), Exponential Double Autoregressive model (Salem & Ahmad, 2018), Gompertz Autoregressive Model (Salem & Ahmad, 2018), and Hyperbolic Tangent Autoregressive model (Mohammed & Mohammad, 2022).

EEAR is characterised by a linear component of the AR(p) structure and a nonlinear component via an exponential transformation. It makes it a dynamic model that can adaptively vary its response based on the larger size of previous observations. As the stability of a solution is a critical prerequisite for forecasting applications or for using the model in economic and financial contexts, the model's geometric ergodicity conditions must be studied carefully, with limit cycles and fixed-point behaviour checked. Since analytical solutions of nonlinear models pose numerous mathematical challenges, the Local Linearization Method has proven to be an appropriate approach for studying the dynamic process in the fixed-point neighbourhood.

This works as follows: we select small neighborhood around the fixed point and replace the equation nonlinear function at it by equivalent linear estimate, making the original system to be known as a linear model, and can be utilized in root analysis, stability test and to characterize the motion at or near the fixed point. Such a mechanism is a powerful tool for confirming the theoretical requirements of nonlinear models, especially when used alongside higher-order Markov chains, as we did previously. The exponentiated exponential distribution that lies at the mathematical foundation of the model imparts several invariants to the transformational process that govern the extent of sharp deviance, the rate of response, and the behavioural variable, as well as the variances of the series parameters. The inclusion of this distribution in the AR(p) framework is one step beyond classical models. It enables the development of a model that addresses temporal dependence and nonlinear probabilistic transformations simultaneously. Due to this dual ability, the model fits particularly well for expressing financial phenomena with nonlinear characteristics, such as precious metal prices, energy supply, and exchange rates. Although the literature has made mention of nonlinear AR models, the works that include both the usage of the exponentiated exponential distribution in the AR(p) architecture, the derivation of the geometric ergodicity condition for fixed points and limit cycles, the application of the local linearization strategy can analyze dynamic behavior, and the real financial transactions to verify the theoretical formulations have been exceptionally limited. Moreover, no applications in which silver prices or related financial series are modelled have been published.

This represents a clear gap that can be filled by adopting a well-rounded, theory- and application-oriented dynamic framework. Real financial series, such as silver prices, exhibit strong nonlinear behaviour that classical linear AR models cannot properly capture. Such an alternative dynamic methodology, which integrates temporal dependence with nonlinear probabilistic transformations, is urgently needed, and stability validation tools for nonlinear models, especially nonstandard probability distributions, also need to be developed. The need for more accurate forecasting instruments is particularly urgent for highly volatile series, and current research doesn't connect the EEAR(p) model to real-world implementations in precious metals pricing. As such, this research makes a truly scientific contribution by connecting theory with application and testing the model's ability to illustrate silver price dynamics and project long-run price levels.

This research will build a computational structure which describes the simulation procedure to predict the stability of the local linearization approximation model of EEAR(p), obtain suitable enough conditions for fixed (non-zero) points to stabilize, apply the model to the experimental data concerning weekly silver prices, test the theoretical stability of the model, and apply the model to obtain optimal series forecasts if the estimated parameters are met.

## 2. The Exponentiated Exponential Distribution

Let  $x$  Represent the random variable, the CDF functions of Exponentiated Exponential with two parameters are (Nadarajah, 2011):

$$F(x, \alpha, \gamma) = (1 - e^{-\gamma x})^\alpha, \quad x \geq 0, \alpha, \gamma > 0 \quad (1)$$

And pdf function has form (Nadarajah, 2011):

$$f(x, \alpha, \gamma) = \alpha \gamma e^{-\gamma x} (1 - e^{-\gamma x})^{\alpha-1}, \quad x \geq 0, \alpha, \gamma > 0 \quad (2)$$

Using MATLAB R2022a, the CDF and PDF functions for the Exponentiated Exponential distribution are plotted for different parameter values, as shown in Figures 1 and 2, respectively.

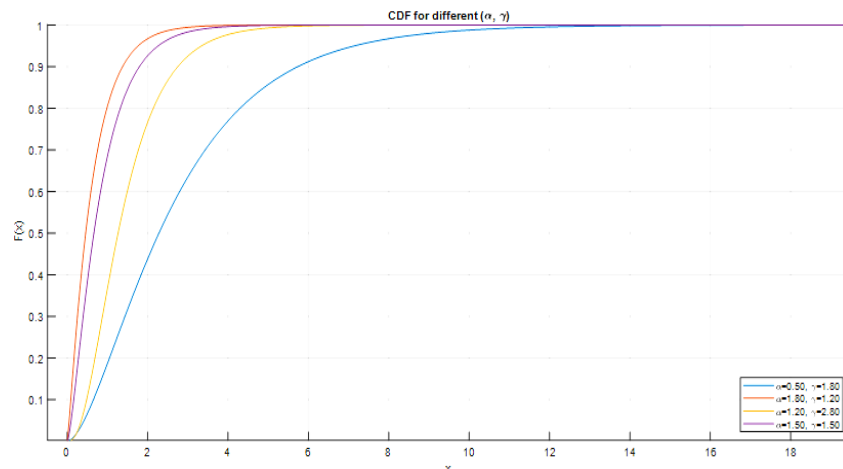


**Figure 1:** CDF Function of Exponentiated Exponential Distribution for Different Parameter Values

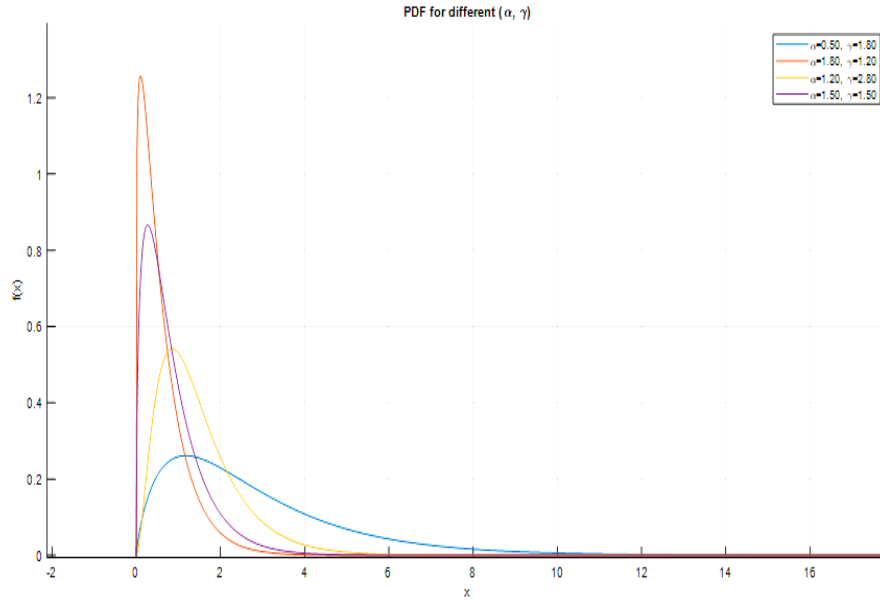


**Figure 2:** PDF Function of Exponentiated Exponential Distribution for Different Parameter Values

### 3. The Autoregressive Exponentiated Exponential Model and its Stability

This study introduces a novel nonlinear time series model that relies on the cumulative distribution function (CDF) of the exponentiated exponential distribution as its basis. The model is constructed through a rigorous analytical framework, with the distribution governed by two parameters: the shape parameter ( $\alpha$ ) and the scale parameter ( $\gamma$ ) (Haggan & Ozaki, 1981). Based on the cumulative distribution function, the general formulation of the Exponentiated Exponential AR(p) Model is expressed as:

$$X_t = \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma X_{t-1}})^\alpha] X_{t-i} + Z_t, Z_t \sim iid N(0, \sigma_z^2) \quad (3)$$

Here  $\{a_i\}$  and  $\{b_i\}$  represent the autoregressive model parameters, for  $i = 1, 2, \dots, p$

The exponentiated exponential distribution function  $F(x, \alpha, \gamma)$  characterized by form (3), possesses a collection of advantageous dynamic properties:

1.  $\lim_{\alpha \rightarrow +\infty} [1 - e^{-\gamma X_{t-1}}]^\alpha = 0$
2.  $\lim_{X_{t-1} \rightarrow 0} [1 - e^{-\gamma X_{t-1}}]^\alpha = 0$
3.  $\lim_{\gamma \rightarrow +\infty} [1 - e^{-\gamma X_{t-1}}]^\alpha = 1$
4.  $\lim_{X_{t-1} \rightarrow +\infty} [1 - e^{-\gamma X_{t-1}}]^\alpha = 1$

The Exponentiated Exponential AR(p) model can be represented in state-space form as follows:

$$X_t = \begin{bmatrix} a_1 & a_2 & \dots & a_{p-1} & a_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} + (1 - e^{-\gamma X_{t-1}})^\alpha \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} + \epsilon_t \quad (4)$$

where the coefficient matrix and vectors are defined accordingly, representing a Markov chain  $\{X_t\}$  with a parameter matrix of dimension  $(p \times p)$  and vectors  $\epsilon_t$  and  $X_{t-1}$  and  $X_{t-1}$  defined as:

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix}, \quad X_{t-1} = \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p} \end{bmatrix}, \quad \epsilon_t = \begin{bmatrix} Z_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where  $\epsilon_t$  represents the white noise innovation vector.

The model can be written in the compact form:

$$X_t = F(X_{t-1}) = T(X_{t-1}) + L(X_{t-1}, \epsilon_t)$$

where  $T(X_{t-1})$  denotes the linear component of the model, while  $L(X_{t-1}, \epsilon_t)$  represents the nonlinear component, defined respectively as:

$$T(X_{t-1}) = \begin{bmatrix} a_1 & a_2 & \dots & a_{p-1} & a_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1}$$

$$L(X_{t-1}, \epsilon_t) = (1 - e^{-\gamma X_{t-1}})^\alpha \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} + \epsilon_t$$

The linear component  $T(X_{t-1})$  satisfies the geometric ergodicity conditions for Markov chains if all eigenvalues of the matrix  $K$  lie strictly inside the unit circle, that is:

$$K = \begin{bmatrix} a_1 & a_2 & \dots & a_{p-1} & a_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

In other words, the eigenvalues of the above matrix can be found as follows:

$$|K - \lambda I| = 0$$

This can be verified by solving the characteristic equation:

$$\lambda^p - \sum_{i=1}^p h_i \lambda^{p-i} = 0 \quad (5)$$

The linear component is stable if all roots of equation (2) lie inside the unit circle, meaning:

$$|\lambda_i| < 1, i = 1, 2, \dots, p$$

First condition: For the nonlinear component  $L(X_{t-1}, \epsilon_t)$  to possess geometric ergodicity properties, the following must hold:

$$\|L(X_{t-1}, \epsilon_t)\| = |(1 - e^{-\gamma X_{t-1}})^\alpha| \left\| \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} \right\|$$

Since:  $\lim_{X_{t-1} \rightarrow +\infty} |(1 - e^{-\gamma X_{t-1}})^\alpha| = 0$

That is:  $0 < |(1 - e^{-\gamma X_{t-1}})^\alpha| \leq 1$

Therefore: there must be  $m$  such that  $0 < m < \infty$  that is:

$$|(1 - e^{-\gamma X_{t-1}})^\alpha| < m e^{-\vartheta t} < \infty, \quad \forall \vartheta > 0$$

for all  $t \geq 0$  and for all  $X_{t-1} \in \mathbb{R}^p$  then we have to get:

$$\|L(X_{t-1}, \epsilon_t)\| \leq m e^{-\vartheta t} \left\| \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} \right\|$$

Consequently, the origin (point zero)  $0 \in \mathbb{R}^p$  constitutes the fixed point of the model, satisfying the fixed-point condition  $L(0) = 0$ . Thus, the first condition is satisfied, establishing the local asymptotic stability condition.

Second condition: The sequence  $\{Z_t\}$  is a stochastic process following  $Z_t \sim iid N(0, \sigma_Z^2)$   $\epsilon_t$  presenting a white noise innovation vector with  $E(\epsilon_t) < C < \infty$ . Therefore, the second condition is satisfied.

Third Condition (Lipschitz Condition): This condition requires verification of the Lipschitz stability criterion  $\mathbb{R}^p$ .



Assume  $L(X_{t-1})$  and  $L(Y_{t-1})$  are arbitrary state vectors  $\mathbb{R}^p$ , then:

$$\|L(X_{t-1}) - L(Y_{t-1})\| = \left\| (1 - e^{-\gamma x_{t-1}})^\alpha \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} X_{t-1} - (1 - e^{-\gamma y_{t-1}})^\alpha \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} Y_{t-1} \right\|$$

Since  $0 < |(1 - e^{-\gamma x_{t-1}})^\alpha| \leq 1$  and  $0 < |(1 - e^{-\gamma y_{t-1}})^\alpha| \leq 1$  there is  $\tau_1, \tau_2 \in \mathbb{R}^p$  such that:

$$\|(1 - e^{-\gamma x_{t-1}})^\alpha\| \leq \tau_1$$

$$\|(1 - e^{-\gamma y_{t-1}})^\alpha\| \leq \tau_2$$

Let  $\tau = \max\{\tau_1, \tau_2\}$  then:

$$\|L(X_{t-1}) - L(Y_{t-1})\| \leq \tau \left\| \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \|X_{t-1} - Y_{t-1}\| \right\|$$

$$\|L(X_{t-1}) - L(Y_{t-1})\| \leq \tau D \|X_{t-1} - Y_{t-1}\|$$

$$\text{where } D = \tau \left\| \begin{bmatrix} b_1 & b_2 & \dots & b_{p-1} & b_p \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right\|$$

Hence, the EEAR(p) model is geometrically ergodic if all roots of the characteristic equation (4) lie strictly inside the unit circle. This provides the necessary condition for model stability, ensuring that the state space of the EEAR(p) model is geometrically ergodic.

To study the specific conditions for the stability of the EEAR(p) model:

$$x_t = \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma x_{t-1}})^\alpha] x_{t-i} + Z_t$$

To identify the non-zero isolated fixed point of the model, set:

$$f(\mu) = \mu$$

$$x_t = x_{t-1} = x_{t-2} \dots x_{t-p} = \mu$$

And eliminate the white-noise innovation effect (since white noise represents negligible random error between 0 and 1; eliminating its effect yields superior forecasting and stability). This yield:

$$\mu = \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma \mu})^\alpha] \mu$$

$$1 = \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma \mu})^\alpha]$$

$$1 = \sum_{i=1}^p a_i + \sum_{i=1}^p b_i(1 - e^{-\gamma \mu})^\alpha$$

$$1 - \sum_{i=1}^p a_i = (1 - e^{-\gamma \mu})^\alpha \sum_{i=1}^p b_i$$

$$(1 - e^{-\gamma \mu})^\alpha = \frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}$$

Assume that:

$$k = \frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i} \quad (6)$$

$$(1 - e^{-\gamma\mu}) = \sqrt[p]{k}$$

$$e^{-\gamma\mu} = 1 - \sqrt[p]{k} \quad (7)$$

$$-\gamma\mu = \ln(1 - \sqrt[p]{k})$$

$$\mu = \frac{-1}{\gamma} \ln(1 - \sqrt[p]{k}) \quad (8)$$

The non-zero singularity  $\mu$  exists and is real (by definition of a constant point it belongs to the real numbers and also exists in the sense that the value is less than infinity) if and only if the following condition is met:

$\ln(1 - \sqrt[p]{k}) < 0$  Taking the exponential function on both sides:

$$1 - \sqrt[p]{k} < 1$$

$$\sqrt[p]{k} > 0$$

$$k > 0$$

That is

$$\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i} > 0$$

Theorem (1): The non-zero singularity  $\mu$  of the EEAR(P) model, if it exists, is stable if and only if all roots of the equation (5). It is located within the unit circle, where

$$h_1 = 1 - \alpha b_1 \left(1 - \sqrt[p]{\frac{1-a_1}{b_1}}\right) \ln \left(1 - \sqrt[p]{\frac{1-a_1}{b_1}}\right) \quad (9)$$

$$h_j = a_j + kb_j \quad j = 2, 3, \dots, p \quad (10)$$

Proof: Let  $\mu$  be a non-zero singular point of the exponential model. Taking an open neighborhood around the point  $\mu$  and a sufficiently small radius, let  $\mu_{t-i}$ ,  $i = 0, 1, 2, \dots, p$  where  $\mu_{t-i}^n \rightarrow 0$  for all  $n \geq 2$ , and  $\mu_{t-i}$  is very small and approaches zero when raised to a large power, we can use the vibratory equations according to the van der Bull equation:

$$x_t = \mu + \mu_t, \quad x_{t-i} = \mu + \mu_{t-i}, \quad i = 1, 2, \dots, p$$

$$\mu_{t-i}^n \rightarrow 0, \quad \forall n \geq 2$$

By substituting  $x_t, x_{t-i}$  and for each  $i = 1, 2, \dots, p$  into the exponential model, and by canceling out the white noise effect  $Z_t$  (i.e., considering  $Z_t = 0$  for each  $t$ ), we obtain:

$$\mu + \mu_t = \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma(\mu + \mu_{t-1})})^\alpha](\mu + \mu_{t-i})$$

Using an approximation of the binomial expansion:

$$(1 - e^{-\gamma(\mu + \mu_{t-1})})^\alpha = 1 - \alpha e^{-\gamma(\mu + \mu_{t-1})}$$

$$\mu + \mu_t = \sum_{i=1}^p [a_i + b_i(1 - \alpha e^{-\gamma\mu} e^{-\gamma\mu_{t-1}})](\mu + \mu_{t-i})$$

$$\mu + \mu_t = \sum_{i=1}^p [a_i + b_i(1 - \alpha e^{-\gamma\mu}(1 - \gamma\mu_{t-1}))](\mu + \mu_{t-i})$$



$$\mu + \mu_t = \sum_{i=1}^p [a_i + b_i(1 - \alpha e^{-\gamma\mu} + \gamma\mu_{t-1}\alpha e^{-\gamma\mu})](\mu + \mu_{t-i})$$

Let  $x = e^{-\gamma\mu}$ , and substitute in the above equation to get:

$$\begin{aligned}\mu + \mu_t &= \sum_{i=1}^p [a_i + b_i(1 - \alpha x + \gamma\alpha\mu_{t-1}x)](\mu + \mu_{t-i}) \\ \mu + \mu_t &= \sum_{i=1}^p [A_i + D_i\mu_{t-1}](\mu + \mu_{t-i})\end{aligned}$$

when  $A_i = a_i + b_i(1 - \alpha x)$ , and  $D_i = b_i\gamma\alpha x$

$$\begin{aligned}\mu + \mu_t &= \mu \sum_{i=1}^p A_i + \mu \sum_{i=1}^p D_i\mu_{t-1} + \sum_{i=1}^p A_i \mu_{t-i} + \mu_{t-1} \sum_{i=1}^p D_i\mu_{t-i} \\ \mu + \mu_t &= \mu \sum_{i=1}^p A_i + \mu \sum_{i=1}^p D_i\mu_{t-1} + \sum_{i=1}^p A_i \mu_{t-i} + \mu_{t-1} \sum_{i=1}^p D_i\mu_{t-i}\end{aligned}$$

But when  $\mu_t\mu_{t-1} \rightarrow 0$  and  $\mu_{t-i}\mu_{t-1} \rightarrow 0$ ,  $i = 1, 2, \dots, p$  we obtain:

$$\mu_{t-1} \sum_{i=1}^p D_i\mu_{t-i} = 0$$

From it we get:

$$\begin{aligned}\mu + \mu_t &= \mu \sum_{i=1}^p A_i + \mu \sum_{i=1}^p D_i\mu_{t-1} + \sum_{i=1}^p A_i \mu_{t-i} \\ \mu_t &= \mu \left( \sum_{i=1}^p A_i - 1 \right) + \mu \sum_{i=1}^p D_i\mu_{t-1} + \sum_{i=1}^p A_i \mu_{t-i} \\ \mu_t &= \mu \left( \sum_{i=1}^p [a_i + b_i(1 - \alpha e^{-\gamma\mu})] - 1 \right) + \mu \sum_{i=1}^p b_i\gamma\alpha e^{-\gamma\mu}\mu_{t-1} + \sum_{i=1}^p [a_i + b_i(1 - \alpha e^{-\gamma\mu})] \mu_{t-i}\end{aligned}$$

From equation 8 and Taylor's expansion, we obtain:

$$\begin{aligned}\mu_t &= \mu \left( \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma\mu})^\alpha] - 1 \right) + \mu \sum_{i=1}^p b_i\gamma\alpha e^{-\gamma\mu}\mu_{t-1} + \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma\mu})^\alpha] \mu_{t-i} \\ \mu_t &= \mu \left( \sum_{i=1}^p [a_i + kb_i] - 1 \right) + \mu \sum_{i=1}^p b_i\gamma\alpha(1 - \sqrt[\alpha]{k})\mu_{t-1} + \sum_{i=1}^p [a_i + kb_i] \mu_{t-i}\end{aligned}$$

From the equation of the original model:

$$\mu = \sum_{i=1}^p [a_i + b_i(1 - e^{-\gamma\mu})^\alpha]\mu$$

Then

$$\mu \left( \sum_{i=1}^p [a_i + kb_i] - 1 \right) = 0$$

Therefore, it will be:

$$\mu_t = \mu \sum_{i=1}^p b_i \gamma \alpha (1 - \sqrt[p]{k}) \mu_{t-1} + \sum_{i=1}^p [a_i + k b_i] \mu_{t-i}$$

By substituting the value of  $\mu$ , we get:

$$\mu_t = \left( \frac{-1}{\gamma} \ln(1 - \sqrt[p]{k}) \right) \sum_{i=1}^p b_i \gamma \alpha (1 - \sqrt[p]{k}) \mu_{t-1} + \sum_{i=1}^p [a_i + k b_i] \mu_{t-i}$$

Substituting the value of  $k$ , we get:

$$\begin{aligned} \mu_t &= -\alpha \ln \left( 1 - \sqrt[p]{\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}} \right) \sum_{i=1}^p b_i \left( 1 - \sqrt[p]{\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}} \right) \mu_{t-1} + \sum_{i=1}^p \left[ a_i + \left( \frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i} \right) b_i \right] \mu_{t-i} \\ \mu_t &= \sum_{i=1}^p \mu_{t-i} - \alpha \ln \left( 1 - \sqrt[p]{\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}} \right) \sum_{i=1}^p b_i \left( 1 - \sqrt[p]{\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}} \right) \mu_{t-1} \end{aligned}$$

Therefore, there are two parts: a linear part which is:

$$\sum_{i=1}^p \mu_{t-i}$$

And a non-linear part, which is:

$$-\alpha \ln \left( 1 - \sqrt[p]{\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}} \right) \sum_{i=1}^p b_i \left( 1 - \sqrt[p]{\frac{1 - \sum_{i=1}^p a_i}{\sum_{i=1}^p b_i}} \right) \mu_{t-1}$$

From equation (5), we get:

$$\therefore \mu_t = h_1 \mu_{t-1} + h_2 \mu_{t-2} + \dots + h_p \mu_{t-p}$$

Which represents the rank difference equation  $p$ :

$$\begin{aligned} h_1 &= 1 - \alpha b_1 \left( 1 - \sqrt[p]{\frac{1 - a_1}{b_1}} \right) \ln \left( 1 - \sqrt[p]{\frac{1 - a_1}{b_1}} \right) \\ h_j &= a_j + k b_j \quad j = 2, 3, \dots, p \end{aligned}$$

The non-zero singularity of the exponentiated exponential model is stable when all roots of the characteristic equation lie within the unit circle, i.e.

$$(|h_i| < 1, i = 1, 2, \dots, p) \quad (11)$$

where  $h_i$  is the root of the characteristic equation ■.

## 4. Real Data Application

The data used in the applied aspect of the study consisted of the weekly average price of silver per ounce in US dollars at the close of the weekly market, from January 1, 2015, to September 18, 2022, totalling 401 observations. This data was obtained from the website (<https://sa.investing.com/commodities/silver>).

It should also be noted that economic indicators are inherently volatile; therefore, forecasts based on them are prone to significant errors and require adjustments to achieve stability (Hauptfleisch et al., 2016).

First, the data analysis process begins by creating a time series. We input the data into MATLAB R2022a and plot it. Figure 3 shows the time series plot with the weekly average price of silver per ounce in US dollars at the close of the weekly market.

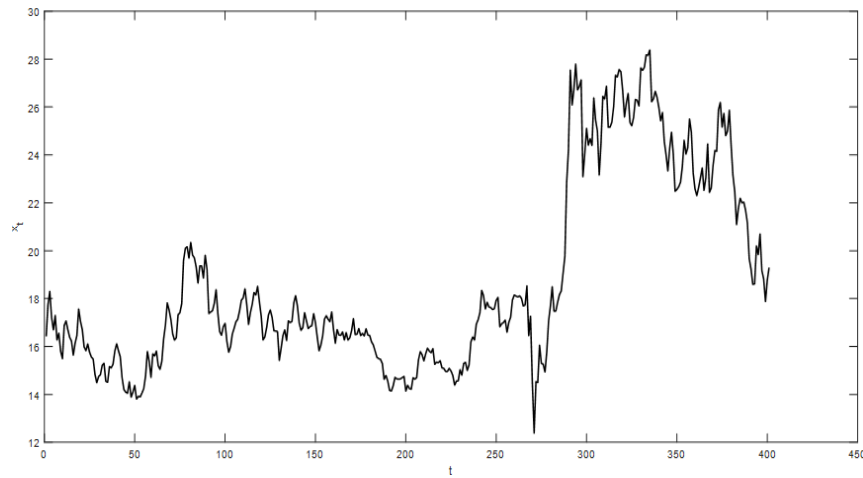


**Figure 3:** Weekly Average Price of Silver Per Ounce in US Dollars at the Close of the Weekly Market

The series does not follow a consistent trend, and there is no clear upward or downward trajectory over the long term. Movement occurs in recurring upward and downward waves (Box et al., 2015). There is significant volatility, with prices rising and falling within wide ranges, reflecting the volatile nature of precious metals markets. The absence of a fixed average around which the series can revolve means it exhibits non-cyclical, non-seasonal, and non-linear components. Furthermore, some rapid price jumps are clearly visible, indicating non-linear behavior, which makes the EEAR model appropriate (Teräsvirta, 2006). The figure clearly demonstrates that the time series requires non-linear models to accommodate this type of volatility, which traditional AR models do not explain (Jury et al., 1976).

Secondly, the time series plotted above is transformed into a Residuals series, which can be expressed mathematically by the following relationship (Tsay, 2005):

$$r_t = \log\left(\frac{x_t}{x_{t-1}}\right)$$

Where  $r_t$  represents the Residuals series and  $x_{t-1}$ ,  $x_t$  represent the observations at times  $t - 1, t$  respectively. The MATLAB instruction for this transformation is  $r_t = \text{price2ret}(x_t)$ , Figure 4 represents the Residuals series.

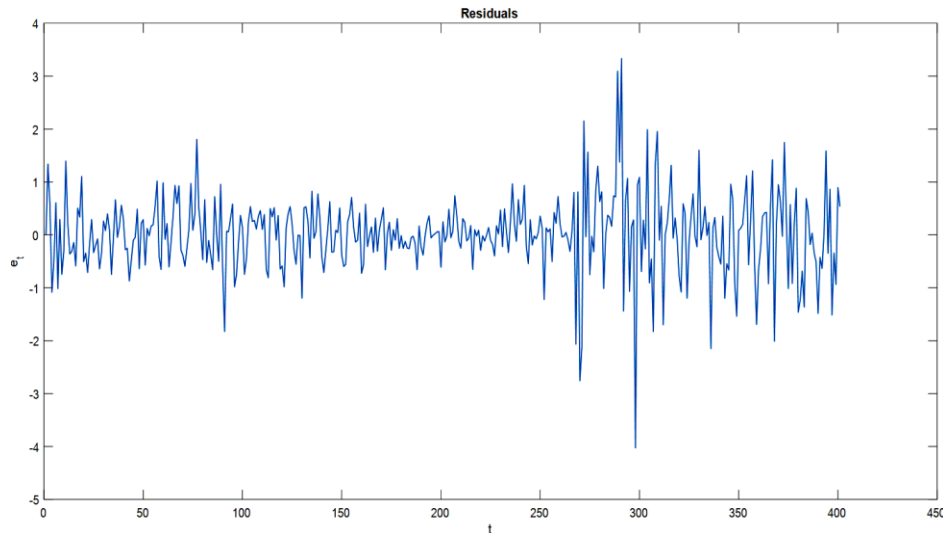


**Figure 4:** Plot the Residuals Series.

The graph shows that the motion oscillates around zero. The residual values are distributed around the zero line without a clear direction, indicating that the difference between successive values is irregular (Akaike, 1974). It also shows that some parts of the residuals exhibit higher variability than others, demonstrating heteroscedasticity. Despite the transformation of the original series into residuals, the nonlinear random behavior remains evident, reinforcing the need for a nonlinear model (Tjøstheim, 1994). Figure 4 shows that the series, after transformation, still contains time-dependent information. This is followed by plotting the total and partial autocorrelation functions (ACF, PACF), respectively, for the residual sequences with 40 lags, as shown in Figure 5.

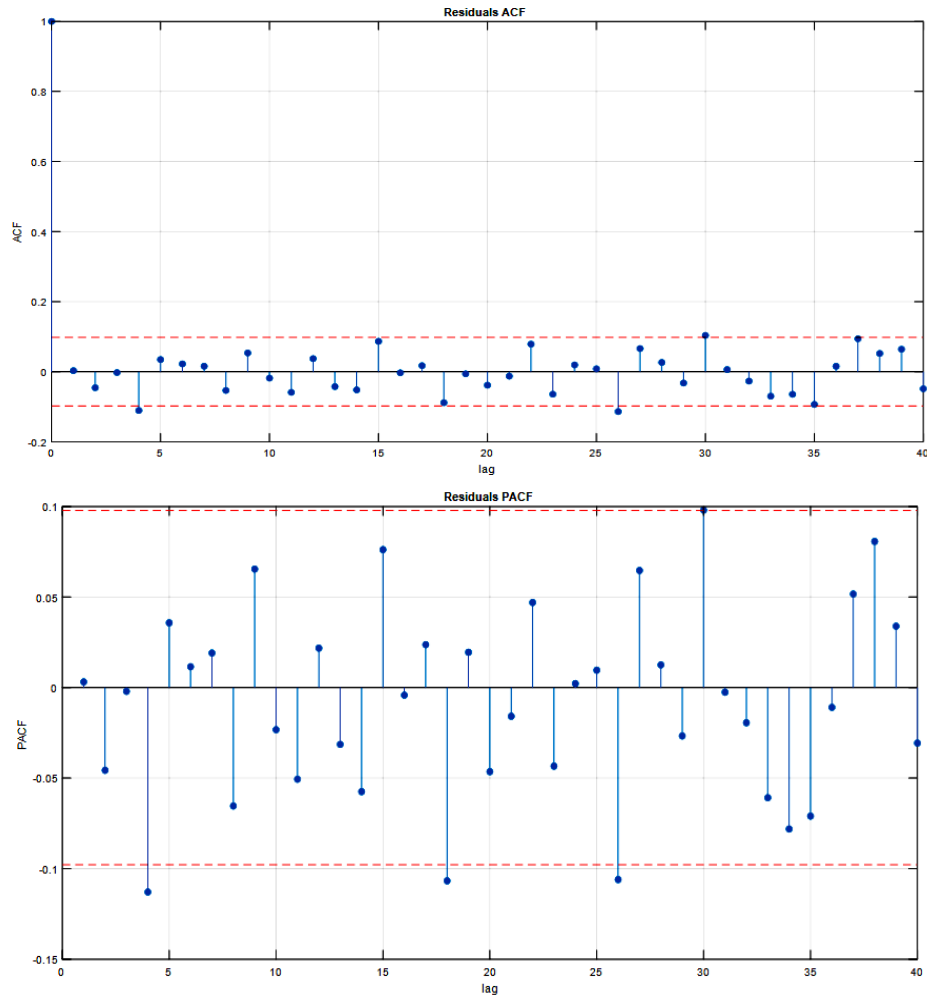


**Figure 5:** Plot ACF, PACF for the Residuals Sequences with 40 Lags

When plotting the ACF, correlation values different from zero at certain lags were observed. This indicates that the residuals are not entirely white, and that a time dependency has not been eliminated. Furthermore, the behavior exhibits oscillation between positive and negative correlations, a sign of nonlinearity. Meanwhile, the PACF shows some values exceeding the confidence interval (Nur Laily et al., 2018; Sahin et al., 2007), indicating direct effects from previous values. This suggests that the model requires a specific  $p$ -order because the residuals are not independent.

Thirdly, in this step, the estimated parameter values for the first ten ranks of the model are found, the variance is calculated, and the information parameters are calculated (Hosseinipoor et al., 2016; Kavitha & Bhuvaneshwari, 2023; Oguntunde et al., 2019). The  $h$  values for all ranks are also found, and the stability condition is verified according to what was proven and explained in equations 9 and 10, and the stability condition is verified in equation 11, as shown in tables 1, 2, and 3, respectively.

**Table 1:** Estimated Parameters, Variance, and  $k$ - Value at Each Order

| EEAR( $p$ ) | $\sum_{i=1}^p a_i$ | $\sum_{i=1}^p b_i$ | $\gamma$ | $\alpha$ | $\sigma^2$ | $k$      |
|-------------|--------------------|--------------------|----------|----------|------------|----------|
| EEAR (1)    | 1.0064             | 0.0068             | 0.5725   | 0.3038   | 0.54461    | 0.941176 |
| EEAR (2)    | 0.9173             | 0.0822             | 3.7730   | 0.3067   | 0.52807    | 1.006083 |
| EEAR (3)    | 96.9286            | 95.9305            | 3.2352   | 0.3193   | 0.5415     | 0.99998  |
| EEAR (4)    | 1.0316             | 0.0381             | 3.4488   | 0.3123   | 0.53548    | 0.829396 |
| EEAR (5)    | 1.0648             | 0.0689             | 7.0211   | 0.3180   | 0.52942    | 0.940493 |
| EEAR (6)    | 0.9998             | 0.0001             | 9.9078   | 0.3000   | 0.53278    | -2       |
| EEAR (7)    | 0.9363             | 0.0634             | 0.8811   | 0.4100   | 0.53147    | 1.004732 |
| EEAR (8)    | 1.0063             | 0.0258             | 0.6600   | 0.2900   | 0.53269    | 0.244186 |
| EEAR (9)    | 1.0036             | 0.0334             | 0.8570   | 0.3500   | 0.52519    | 0.10778  |
| EEAR (10)   | 1.0417             | 0.1349             | 1.1300   | 0.4100   | 0.54367    | 0.307635 |

**Table 2:** Evaluate Information Criteria at Each Order

| EEAR(p)   | L        | AIC     | CAIC    | BIC     | HQIC    |
|-----------|----------|---------|---------|---------|---------|
| EEAR (1)  | 122.145  | 236.29  | 241.19  | 220.304 | 229.96  |
| EEAR (2)  | 129.0245 | 246.049 | 251.899 | 226.079 | 238.142 |
| EEAR (3)  | 124.6825 | 233.365 | 240.155 | 209.416 | 223.881 |
| EEAR (4)  | 127.606  | 235.212 | 242.931 | 207.289 | 224.153 |
| EEAR (5)  | 130.5585 | 237.117 | 245.756 | 205.226 | 224.485 |
| EEAR (6)  | 129.985  | 231.97  | 241.516 | 196.114 | 217.766 |
| EEAR (7)  | 131.1575 | 230.315 | 240.759 | 190.501 | 214.542 |
| EEAR (8)  | 131.3885 | 226.777 | 238.109 | 183.009 | 209.436 |
| EEAR (9)  | 134.867  | 229.734 | 241.942 | 182.018 | 210.827 |
| EEAR (10) | 128.7495 | 213.499 | 226.573 | 161.84  | 193.027 |

**Table 3:** Evaluate the  $h_t$ - value at Each Order

| EEAR(p)   | $h_1$                | $h_2$   | $h_3$   | $h_4$   | $h_5$  | $h_6$   | $h_7$   | $h_8$   | $h_9$  | $h_{10}$ | Stability or not |
|-----------|----------------------|---------|---------|---------|--------|---------|---------|---------|--------|----------|------------------|
| EEAR (1)  | 0.9986               |         |         |         |        |         |         |         |        |          | Yes              |
| EEAR (2)  | -1.8110              | -0.0342 |         |         |        |         |         |         |        |          | No               |
| EEAR (3)  | $4.7542 \times 10^9$ | 32.3540 | 42.5721 |         |        |         |         |         |        |          | No               |
| EEAR (4)  | 0.9662               | -0.0462 | 0.0645  | 0.0048  |        |         |         |         |        |          | Yes              |
| EEAR (5)  | 0.8221               | -0.0411 | 0.0491  | -0.1077 | 0.1605 |         |         |         |        |          | Yes              |
| EEAR (6)  | 2.3771               | -0.0380 | 0.0528  | -0.0689 | 0.0562 | -0.0613 |         |         |        |          | No               |
| EEAR (7)  | 1.0205               | -0.0992 | -0.0354 | -0.0237 | 0.1374 | -0.0370 | 0.0238  |         |        |          | No               |
| EEAR (8)  | 0.9938               | -0.0107 | 0.0381  | -0.0991 | 0.1552 | -0.0729 | -0.0019 | -0.0151 |        |          | Yes              |
| EEAR (9)  | 1.0113               | -0.0994 | 0.0860  | -0.1180 | 0.1346 | -0.0479 | 0.0557  | -0.0990 | 0.0571 |          | No               |
| EEAR (10) | 0.9877               | 0.0763  | -0.0989 | 0.0175  | 0.1406 | -0.0822 | 0.1265  | -0.1962 | 0.1983 | -0.1207  | Yes              |

The results obtained from the three preceding tables, derived from the multi-order models of the EEAR(p) series, show a clear variation in the statistical and behavioral performance of each model order. Upon analyzing the parameters of each model, the lower-order models, such as  $p = 1$  and  $p = 2$ , appeared to have a simple structure and few parameters, which was reflected in favorable information criteria in some cases. However, this advantage was insufficient when the dynamic stability condition was introduced, namely, examining the values of the roots and their position within the unit circle, in addition to analyzing the h-coordinates derived from the theoretical structure of the exponentiated exponential AR model.

The first-order model provided good initial indicators in terms of its AIC value, which ranked among the best in the initial comparison. However, the stability test revealed less stable behavior. This caused the model to exhibit borderline behavior and high sensitivity to fluctuations. The second-order model benefited from the addition of a second AR coefficient, but the resulting  $h$  - values indicated significant instability. Its polynomial roots also showed a clear departure from unity, weakening its long-term reliability despite improvements in some information parameters.

The  $p = 3$  The model exhibited a significant inflation of coefficient values, directly resulting in an explosion of  $h$  values and difficulty in guaranteeing a stable fixed point. Although its AIC was within acceptable limits, dynamic stability was never achieved, rendering this model unsuitable for long-term dependency-based time series. Moving to higher-order models,  $p = 4$  and  $p = 5$  demonstrated a better balance between theoretical structure and information parameters. The  $p = 4$  model exhibited stable polynomial roots and relatively ordered AR coefficients. However, its AIC and BIC values, while acceptable, did not reach the level required to make it superior to other models. The  $p = 5$  model stood out as the only one that combined clear stability in  $h$  values with complete satisfaction of the root condition. It also had the best AIC among all stable models. Furthermore, its  $\text{Sigma}^2$  value was lower than those of other orders, indicating greater effectiveness in explaining time series variance. This model maintained a delicate balance between the number of parameters and statistical performance, without suffering from the inflation of coefficient values observed in the third order.

Higher-order models such as  $p = 7$  and  $p = 8$  achieved varying degrees of stability, and some of their dynamic readings were acceptable. However, their information parameters (particularly BIC and HQIC) weakened their evaluation due to the large number of coefficients. Through this integrated analysis of statistical and behavioral criteria, it becomes clear that the  $p = 5$  model represents the optimal choice. It exhibited the best AIC value among the stable models, and achieved the smallest.  $\sigma^2$  error, and met the stability condition through consistent  $h$  values and the retention of roots within the unit circle. Thus, EEAR (5) is the most balanced and accurate model for describing the dynamic behavior of the time series while maintaining an appropriate level of complexity without an excessive number of parameters.

Therefore, the equation for the best model (EEAR (5)) is in the form:

$$x_t = 1.0040x_{t-1} - 0.0411x_{t-2} + 0.0491x_{t-3} - 0.1077x_{t-4} + 0.1605x_{t-5} \\ + (0.0056x_{t-1} - 0.0069x_{t-2} - 0.00201x_{t-3} - 0.0104x_{t-4} - 0.0552x_{t-5})(1 - e^{-7.0211x_{t-1}3})^{0.3180}$$

This equation represents the model residual variance  $\sigma_z^2 = 0.52942$ . Figure (6 shows a plot of the model curve in the  $x_t, x_{t-1}$  Plane. We will notice that there are points in the plot, and these points are the data for the time series. Figure 7 contains the normal probability plot of the EEAR (5) model residuals, and the points indicated on the plot represent the values of the model residual series. The residuals are the differences between the true values of the series and the values predicted by the model, while Figure 8 shows the Q-Q plot of the residuals of the EEAR (5) model.

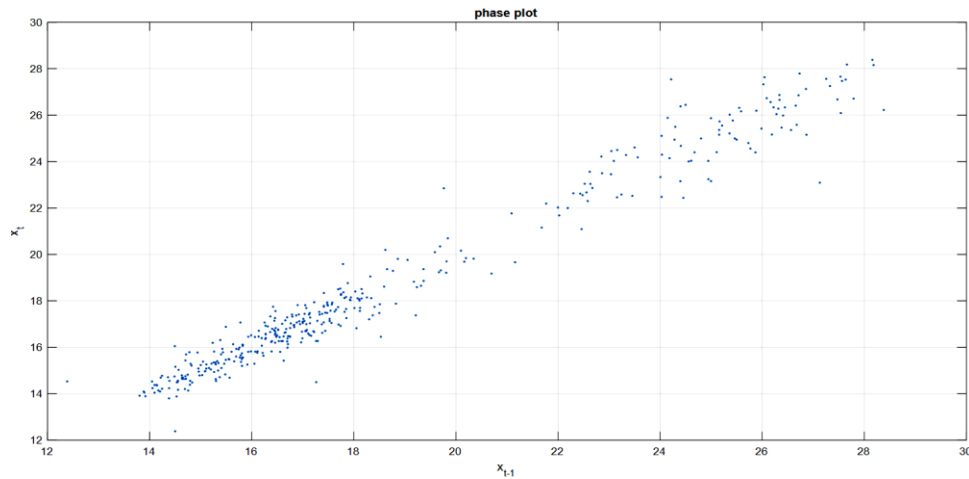


**Figure 6:** Plot of the Model Curve in the  $x_t, x_{t-1}$ - Plane

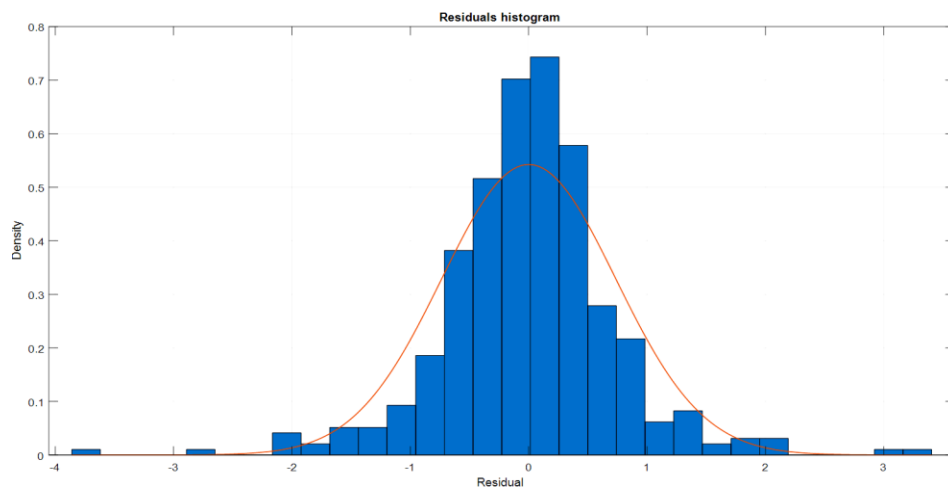


**Figure 7:** The Normal Probability Plot of the EEAR (5) Model Residuals

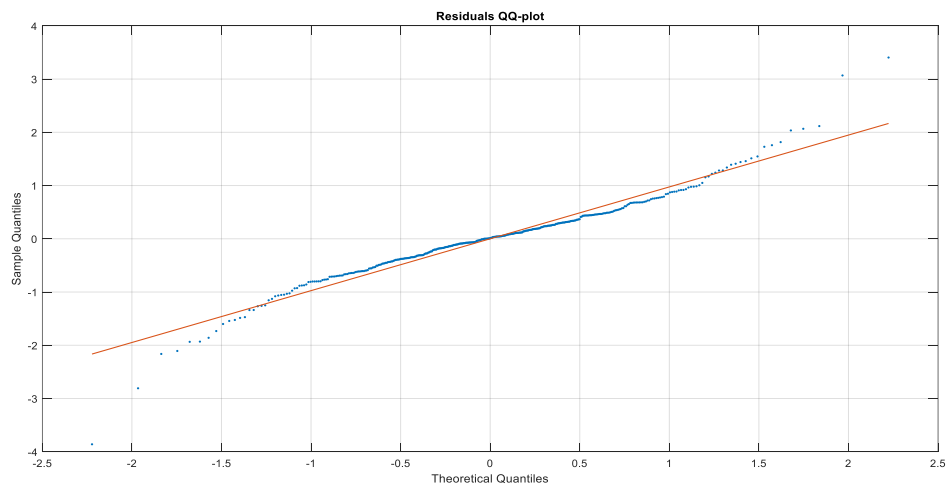


**Figure 8:** Q-Q Plot of the Residuals of the EEAR (5) Model



Figure 9 shows the plotting of the EEAR (5) model paths for different initial values, which shows all paths approaching the non-zero singularity of the model, while Figure 10 shows the path and value of the non-zero singularity obtained from applying Equation 8, as shown in the aforementioned figure.

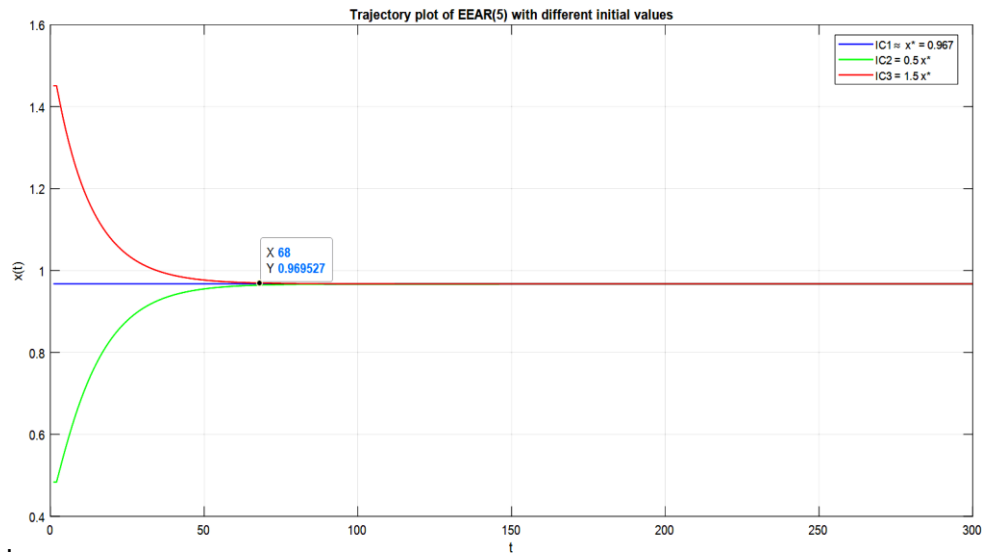


**Figure 9:** Plotting of the EEAR (5) Model Paths for Different Initial Values

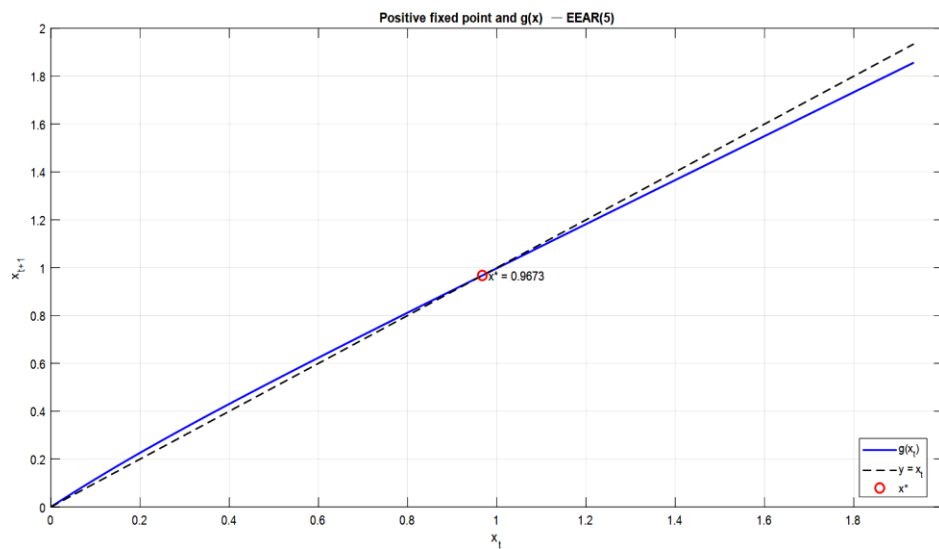


**Figure 10:** The Path and Value of the Non-Zero Singularity

To perform the dynamic analysis of the nonlinear model, we use the trajectories of the EEAR (5) model at varying initial values, as illustrated in Figure 9. The value starts from different initial levels which we know evolve, as the figure above shows. By tracking the movements of each of these trajectories it is evident that all trajectories (in spite of their differing beginning points) end up converging to the single non-zero point the model depicts. This behavior shows a simple characteristic known as local stability toward the central point. The remarkable element in the illustration is that the trajectories do not diverge too far, nor do they have explosive cycles and uncontrolled trajectories, they instead converge to the same level, indicating that the model is reactive and limited to long-range constant values. This is in direct correlation with the stability test output, where the roots of the characteristic equation are located within the fifth order unity circle, allowing the model to internalize the initial perturbations and gently return to the equilibrium region. This quantitative observation thus serves as a practical reinforcement to the theoretical component of the model stability. An estimate of the trajectory of the same non-zero singularity, based on Equation (8) in the theoretical part of the research, is provided in Figure 10.

Figure 10 shows the comparison of the singularity as a theoretical value to an actual motion on this trajectory through the model. The singularity obtained from the mathematical equation by the model has literally been mapped to the point of approach for time paths in Figure 9. This correspondence between the theoretical and practical values shows the model is

able to remain stable for not only polynomial roots, but also the actual time behavior of the sequence. This shows that the EEAR (5) model has an actual equilibrium that can be dynamically analyzed, and that the system, despite the initial fluctuations, eventually leads to a stable value represented by the equilibrium of a real-time process. This confers greater confidence in the predictive ability of our model, as it doesn't suffer from runaways or large, unstable oscillations.

## 5. Conclusions

In theory, the EEAR(p) model was created by integrating the linear structure of the autoregressive model and probabilistic transformation of an exponential distribution which could be computed with weighted exponentials. Dynamic stability conditions were developed on the basis of the following three terms: the roots of the linear segment, the presence of a non-zero singularity, and the Lipschitz condition of the nonlinear segment. Mathematical experimentation also showed that the stability of the model is supported by the fact that all roots of the characteristic equation need to lie within the unit circle and the nonlinear section of the function must be limited by the result of the exponential transformation to avoid numerical bursts. The non-zero singularity equation resulted accurately, and the theoretic study showed this model has peculiar behavior which combines probabilistic elasticity with time dependence, a feature not reflected in traditional linear models. Realistically, the model was implemented to weekly silver price data, which is very random and nonlinear. The first series analyses indicated that there were non-standard upward/downward waves and no long-term trends, making linear models unsuitable. Based on residuals and autocorrelation functions, it can be stated that in such a time series the temporal feature of the solution needs a flexible model. As we compared order 1 to order 10 models, EEAR (5) was found to be the balanced model which demonstrated to provide the best results for information criteria such as AIC and satisfy the dynamic stability conditions and roots both the relationships in unity circle and reduced the variance appreciably. Additionally, the graphs (including Figures 9 and 10) proved that all model paths converge to the same singular point, showing the practical, and not only theoretical, stability of the model. As a consequence, the model was able to accurately model the actual behavior of the series and to model its future direction very well.

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