

Analysis and Exact Solution of the Fractional Bagley-Torvik Equation Using Symmetry Methods

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ABSTRACT

In this paper, we perform a Lie symmetry analysis of time-fractional Bagley–Torvik (TF-BT) equation by employing Caputo fractional derivatives. The TF-BT equation describes the time response of viscoelastic materials with taking into account memory effects. The Lie symmetry method is used to find the symmetry generators, invariant solutions, symmetry reductions, and conservation laws. Symmetry reductions provide exact similarity solutions, where the admitted symmetry group follows from a prolongation of fractional derivative structure. Even without a classical Lagrangian, conservation laws can then be derived from Ibragimov's theorem. Lie symmetry analysis is a useful technique for fractional differential equations, as demonstrated by the analytical solutions of the TF-BT problem that are obtained and fulfill this equation.

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1. Introduction

FDEs are a powerful, novel class of mathematical models that, due to their higher-order nature, can more effectively capture the complex physical behaviour of materials than conventional integer-order differential equations. FDEs are non-local, including memory directly in the constitutive relation; they preserve history-dependent behaviour and temporal dispersion relevant to a versatile class of engineering and physical systems, in contrast to classical formulations. This inherent property makes fractional models especially effective for problems that traditional methods cannot address, as they overlook hereditary effects, long-range interactions, or anomalous diffusion mechanisms. As a result, FDEs offer a powerful and versatile tool for accurately modelling and analysing the dynamics of materials and systems in which historical states have a substantial impact on future behaviour.

This lack of theoretical guidance has severely limited the study of fractional differential equations (FDEs). Sophus Lie developed the Lie group transformation methodology, which provides a systematic algorithmic method for accurately solving differential equations. Such an approach is particularly beneficial for tackling systems of PDEs or FDEs.

In the last ten years Lie symmetry analysis has been employed on integro-differential and linear differential equations, see for instance (Bluman & Kumei, 2013; Fushchich et al., 2013; Ibragimov & Kovalev, 2009; Meleshko, 2006). Gazizov et al. (2007) established the first application of Lie symmetry theory to FDEs. Many other researchers have studied the FDEs and potential improvements were also reported Solving FDEs (Djordjevic & Atanackovic, 2008; Wang et al., 2013).

Finding exact solutions is one of the main advantages of Lie point symmetry. Exact solutions usually characterize the leading or asymptotic behavior of more general families of solutions, and they can use to compare numerical methods and experimental validations. Identification of the Lie symmetry group corresponding to a differential system is an effective tool for order reduction of equations, especially for ordinary differential equations (ODEs) in a single parameter, since a reduction by one degree can always be obtained. This results in a single quadrature allowing for exact solutions to be obtained. In addition, Lie symmetries enable the classification of differential equations by the form of arbitrary parameter or functional structures.

A second use of Lie group theory is the creation of conservation rules, which are used to assess the existence, uniqueness, as well as stability of solutions (Bluman & Anco, 2002).

The Bagley–Torvik (BT) equation, first proposed to model the dynamics of a rigid body in a viscoelastic liquid. To simulate systems with memory and inherited characteristics, it extends the classical form by replacing the first-order time

derivative with a fractional-order Caputo derivative.

We study the following TF-BT equation:

$$aD_t^2 u(t, x) + b {}^C D_t^\alpha u(t, x) + cu(t, x) = 0 \quad 0 < \alpha < 1 \quad (1)$$

where a , b , and c are constants, $D_t^2 u$ is the second-order classical time derivative, t and x are independent variables, and ${}^C D_t^\alpha u$ is the fractional derivative of Caputo time.

The time-fractional Bagley–Torvik equation has been studied analytically and numerically using Laplace transforms, spectral methods, operational matrices, and other tools. For instance, [Shlash Mohammad et al. \(2025\)](#) proposed a numerical scheme for solving the Bagley–Torvik (BT) equation. In this study, the collocation method is applied using Müntz–Legendre wavelets (MLWs). The unknown solution is then approximated on the MLW basis. The resulting residual is expressed in terms of the operational matrices for the integer and Caputo fractional derivatives by substituting this approximation into the BT equation. In some situations, perfect solutions can be obtained due to the scheme's structure and the adaptable design of Müntz–Legendre wavelets. In their evaluation of the BT equation solution using the integral transform approach, [Zafar et al. \(2020\)](#) employ the Caputo fractional derivative for accurate physical modeling. The answer can be formulated analytically using the GGG function. Parametric studies will point out the impact of fractional orders on system dynamics. In [Zeen El Deen et al. \(2025\)](#), a numerical scheme is proposed to solve the FBTE. In this context, the present authors (Mohamed Ramadan Zeen El Deen and colleagues) have also adopted and integrated fourth-kind Chebyshev polynomials into the intuitive framework of the operational tau method. A primary improvement is the ability to efficiently address non-homogeneous boundary conditions by systematically transforming them into equivalent homogeneous forms. The results illustrate the effectiveness of the Chebyshev polynomial operational tau approach in dealing with the complexities associated with the Fractional Bagley–Torvik problem.

This work is aimed at the exact analysis of the time-fractional Bagley–Torvik equation (FBTE), wherein we analyse the performance of the improved analytical Lie symmetry method to derive accurate solutions, study the impact of fractional parameters, and unveil underlying symmetries. Applications of the FBTE span fields such as viscoelasticity, biomechanics, vibration analysis, and fractional control systems.

The current work is structured and organized as follows. Section 2 reviews several key concepts and traits of fractional calculus. The procedure for determining the Lie point symmetries of the TF-BT equation is explained in Section 3. The application of these Lie symmetries to create new invariant solutions utilizing the similarity reduction approach is the main topic of the next section. In Section 5, conservation rules for the FBTE are developed using Lie symmetry generators. The last part presents the results.

2. Preliminaries and Some Definitions

This section presents the essential concepts and theorems of fractional calculus required for the subsequent analysis. For more details, see references ([Kilbas et al., 2006](#); [Podlubny, 1999](#)).

Definition 2.1: For integrable function $f(t) \in (L_1[a, b], \mathbb{R})$, the left sided and right sided fractional Riemann-Liouville (RL) integrals of the order of $\alpha > 0$ are defined as

$${}_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a, \quad (2)$$

$${}_t I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t < b, \quad (3)$$

where Γ represents the Gamma function ([Podlubny, 1999](#)).

The following are some characteristics of the operator I^α . For $\alpha, \gamma \geq 0$ and $\beta \geq -1$ we have:

$$I_t^\alpha I_t^\beta f(t) = I_t^{\alpha+\beta} f(t), I_t^0 f(t) = f(t), I_t^\alpha I_t^\gamma f(t) = I_t^\gamma I_t^\alpha f(t), I_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

Definition 2.2: Given a function $f(t)$ defined on the interval $[a, b]$, the left-hand and right-hand fractional RL derivatives of order α are as follows:

$${}_a D_t^\alpha f(t) = D_t^n ({}_a I_t^{n-\alpha} f(t)) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (4)$$

$${}_t D_b^\alpha f(t) = (-1)^n D_t^n ({}_t I_b^{n-\alpha} f(t)) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau, \quad (5)$$

where $n = [\alpha] + 1$.

Here is a list of some characteristics of RL derivatives. For $\alpha, \beta, \gamma > 0$ we have:

- $D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}$, where t^β is α differentiable,
- $D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}$, where t^β is α differentiable,
- $D_t^\gamma C = \frac{ct^{-\gamma}}{\Gamma(1-\gamma)}$, C is a constant,
- $D_t^\beta [h(t)l(t)] = \sum_{m=0}^{\infty} \binom{\beta}{m} D_t^{\beta-m} h(t) D_t^m l(t)$, $\beta > 0$.

The generalized Leibnitz rule is the last characteristic, where $\binom{\beta}{m} = \frac{(-1)^{m-1} \beta \Gamma(m-\beta)}{\Gamma(1-\beta) \Gamma(m+1)}$.

Definition 2.3: If $n = [\alpha] + 1$, the Caputo fractional derivative operators of order $\alpha \in \mathbb{R}^+$ on both the left and right sides are defined by

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (6)$$

$${}_t^C D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^n(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau. \quad (7)$$

3. Analysis of Symmetry in the Time Fractional Bagley-Torvik (BT) Equations

To utilise FDEs, especially the time-fractional Bagley-Torvik (BT) equation, the current work first conducts a Lie symmetry analysis. In the first part, we give the infinitesimal form of the one-parameter Lie group of point transformations and show how it can be used to find the Lie point symmetries of FDEs. Next, the equivalent infinitesimal generator is constructed. We employ the proper prolongation sensitive to both integer-order & fractional derivatives for this FDE imposed with an invariance based on the trajectory of physical laws. This leads to an overdetermined system of partial differential equations with FDEs. Using symbolic computing tools like the FracSym package (Jefferson & Carminati, 2014), the fractional system (Djordjevic & Atanackovic, 2008) may be solved algebraically or numerically to determine the system's Lie point symmetries.

Consider the following equation

$$F = au_{tt} + b^C D_t^\alpha u + cu = 0. \quad (8)$$

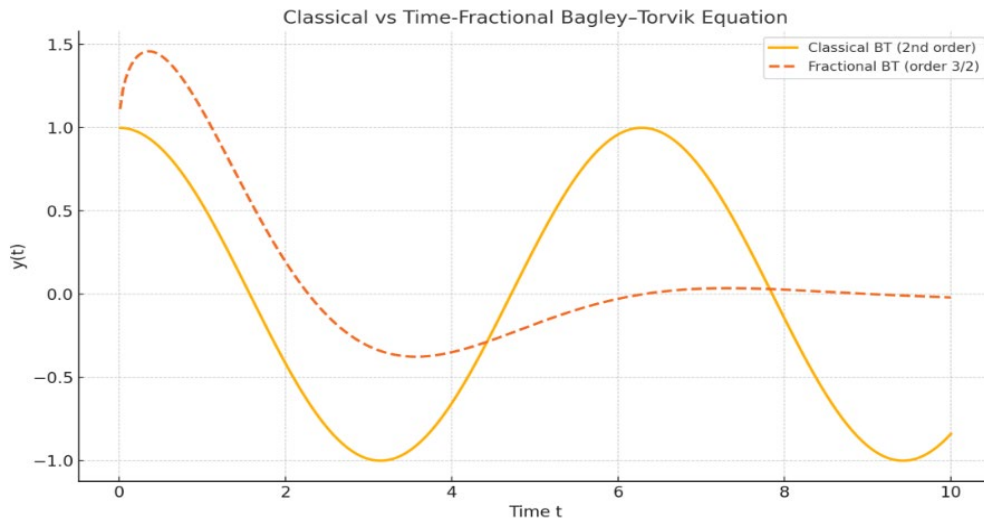


Figure 1: Here is the Plot Comparing the Classical and Time-Fractional Bagley–Torvik Solutions:  Blue Curve: Classical Solution $y(t)=\cos(t)$, Which Oscillates with Constant Amplitude.  Orange Dashed Curve: Fractional Solution using D_t^α , Which Shows Slower Decay and Memory-Driven Damping — a Hallmark of Fractional-Order Systems

According to Lie group theory, if a one-parameter Lie group of point transformations is accepted in Eq. (8), then the following infinitesimal Lie point transformations are satisfied:

$$\begin{aligned}\bar{t} &= t + \epsilon\tau(t, x, u) + o(\epsilon^2), \\ \bar{x} &= x + \epsilon\xi(t, x, u) + o(\epsilon^2), \\ \bar{u} &= u + \epsilon\eta(t, x, u) + o(\epsilon^2).\end{aligned}\quad (9)$$

We refer to the collection of all these transformations as a continuous Lie group G if the structure of equation (8) is maintained in the new variables $\bar{t}, \bar{x}$, and $\bar{u}$. In the aforementioned transformations, ϵ is a continuous parameter. These transformations' corresponding infinitesimal generator is provided by

$$V = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}, \quad (10)$$

Where we have

$$\left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon=0} = \tau(t, x, u), \quad \left. \frac{d\bar{x}}{d\epsilon} \right|_{\epsilon=0} = \xi(t, x, u), \quad \left. \frac{d\bar{u}}{d\epsilon} \right|_{\epsilon=0} = \eta(t, x, u). \quad (11)$$

Additionally, to determine the lower limit of the integral in (4) and preserve the structure of the RL fractional derivative during transformation (9), the equation $t=0$ must be invariant with respect to Lie point transformations (Gazizov et al., 2007).

Therefore, one can results

$$\tau(t, x, u)|_{t=0} = 0. \quad (12)$$

The transformation group (9) & vector fields (10) are accepted by Eq. (8) due to the invariance condition if and only if the extended vector field $\text{Pr}^{(\alpha,2)}V$ disappears Eq. (8) on its solution, indicating that

$$\text{Pr}^{(\alpha,2)}V(F) \Big|_{F=0} = 0. \quad (13)$$

This equation defines all infinitesimal symmetries of Eq. (8); thus, we refer to it as the determining equation. By retaining essential terms, the prolongation of the operator $\text{Pr}^{(\alpha,2)}V$ assumes the following form:

$$\text{Pr}^{(\alpha,2)}V = V + \eta^t \frac{\partial}{\partial u_t} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \eta^{(\alpha,t)} \frac{\partial}{\partial {}^C D_t^\alpha u}. \quad (14)$$

Thus, one can get the following the invariance criterion

$$\text{Pr}^{(\alpha,2)}V[au_{tt} + b {}^C D_t^\alpha u + cu]|_{\text{Eq}} = 0. \quad (15)$$

The Second-order classical prolongation formula for partial differential equations is expressed:

$$\begin{aligned}\eta^{tt} &= D_t^2(\eta - \tau u_t - \xi u_x) + \tau u_{ttt} + \xi u_{ttt} \\ &= \eta_{xx} + (2\eta_{xu} - \xi_{xx} - 2\tau_{xu}u_t - 3\xi_u u_{xx})u_x \\ &\quad - \tau_{xx}u_t + (\eta_{uu} - \tau_{uu}u_t - 2\xi_{xu})u_x^2 - 2(\tau_x + \tau_u u_x)u_{xt} \\ &\quad - (\tau_u u_t - \eta_u + 2\xi_x)u_{xx} - \xi_{uu}u_x^3.\end{aligned}$$

The time fractional Caputo derivative's extended infinitesimal $\eta^{(\alpha,t)}$ may be found using the following formula.

$$\eta^{(\alpha,t)} = {}^C D_t^\alpha (\eta - \tau u_t - \xi u_x) + \tau \frac{d}{dt} ({}^C D_t^\alpha u).$$

$$\text{Pr}^{(\alpha,2)}V[au_{tt} + b {}^C D_t^\alpha u + cu] = a\eta^{tt} + b\text{Pr}^{(\alpha)}V({}^C D_t^\alpha u) + c\eta = 0.$$

Substituting $Q = \eta - \tau u_t - \xi u_x$, and express η^{tt} , ${}^C D_t^\alpha(Q)$, etc. and expanding and equating all coefficients of independent derivatives (e.g. u_t, u_x, u) to zero yields the determining system of PDEs for τ, ξ, η .

$$\tau(t, x, u) = c_1 t + c_2, \quad \xi(t, x, u) = c_3 x + c_4, \quad \eta(t, x, u) = (c_3 - \alpha c_1)u. \quad (16)$$

where $c_i (i = 1, 2, 3, 4)$ and $F_i (i = 1, 2, 3)$ are arbitrary constants and arbitrary function respectively. By regarding the condition (0) = 0, every admitted operator of equation (1) can be presented as:

$$V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = t\partial_t + \frac{\alpha}{2}x\partial_x - \alpha u\partial_u. \quad (17)$$

Thus, the dimension of Lie algebra, admitted by the equation fractional (BT) (1) is $L_3 \oplus L_\infty$. We may use the MAPLE

symbolic computing platform to compute the symmetries as finding symmetries for FDEs and long computations is algorithmic. The infinitesimal generators and equations for the symmetries of FDEs may be found using the symmetry program FracSym (Jefferson & Carminati, 2014) in combination with the MAPLE symmetry packages DES-OLVII (Vu et al., 2012) & ASP (Jefferson & Carminati, 2013).

4. Reductions in Similarity and Precise Solutions

In this section, we describe the similarity-reduction approach and extend it to solve Eq. (1) exactly. The foundation of this approach is the discovery of reductions of PDEs & EDEs to systems with fewer independent variables. We call u an invariant solution if and only if it is equal to zero after applying infinitesimal generators of symmetries of the equation (see (Djordjevic & Atanackovic, 2008; Singla & Gupta, 2017)). Now we present group-invariant solutions of the fractional Eq. (1) by using its symmetries.

Use the scaling generator: $V_3 = t \partial_t + \frac{\alpha}{2} x \partial_x - \alpha u \partial_u$

We solve the characteristic equations:

$$\frac{dt}{t} = \frac{dx}{\frac{\alpha}{2}x} = \frac{du}{-\alpha u}, \quad (18)$$

$$\frac{dt}{t} = \frac{dx}{\frac{\alpha}{2}x} \Rightarrow \frac{dt}{t} = \frac{2dx}{\alpha x} \Rightarrow x = \zeta t^{\alpha/2}. \quad (19)$$

Let $\zeta = xt^{-\alpha/2}$ be the similarity variable. Then we have

$$\frac{dt}{t} = \frac{du}{-\alpha u} \Rightarrow u = t^{-\alpha} \varphi(\zeta).$$

which results in the following similarity variables

$$u(t, x) = t^{-\alpha} \varphi(\zeta), \quad \zeta = xt^{-\alpha/2}. \quad (20)$$

Thus, one can conclude

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\alpha t^{-\alpha-1} \varphi(\zeta) + t^{-\alpha} \dot{\varphi}(\zeta) \frac{d\zeta}{dt}, \\ \frac{d\zeta}{dt} &= \frac{d}{dt} \left(\frac{x}{t^{\alpha/2}} \right) = -\frac{\alpha}{2} \frac{x}{t^{\alpha/2+1}} = -\frac{\alpha}{2} \frac{\zeta}{t}, \\ \Rightarrow \frac{\partial u}{\partial t} &= t^{-\alpha-1} \left(-\alpha \varphi(\zeta) - \frac{\alpha}{2} \zeta \dot{\varphi}(\zeta) \right), \\ \frac{\partial^2 u}{\partial t^2} &= t^{-\alpha-2} \left[\alpha(\alpha+1) \varphi(\zeta) + \alpha(\alpha+1) \frac{\zeta}{2} \dot{\varphi}(\zeta) + \frac{\alpha^2}{4} \zeta^2 \ddot{\varphi}(\zeta) \right]. \end{aligned} \quad (21)$$

After substituting expression (21) into Eq. (1), we have

$${}^c D_t^\alpha u(t, x) = t^{-2\alpha} [C_1 \varphi(\zeta) + C_2 \zeta \dot{\varphi}(\zeta) + C_3 \zeta^2 \ddot{\varphi}(\zeta)], \quad (22)$$

where C_1, C_2, C_3 are constants depending on α .

Definition 4.1 The fractional integral operator Erdős-Kober is presented as

$$(\kappa_\delta^{\tau, \alpha} g)(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^\infty (r-1)^{\alpha-1} r^{-(\tau+\alpha)} g(zr^{\frac{1}{\delta}}) dr, & \alpha > 0, \\ g(z), & \alpha = 0. \end{cases} \quad (23)$$

Using the Erdős-Kober (EK) fractional integral operator, the extended left-handed EK fractional differential operator may be described as follows:

$$(\rho_\delta^{\tau, \alpha} g)(z) = \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\delta} z \frac{\partial}{\partial z} \right) (\kappa_\delta^{\tau+\alpha, n-\alpha} g)(z), \quad (24)$$

where $z > 0, \alpha > 0, \delta > 0$ and

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}$$

Conversely, based on the EK fractional integral operator formulation provided in (23), we obtain

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n} \left[t^{n-2\alpha} \left(\kappa_2^{\frac{1-\alpha, n-\alpha}{\delta}} \varphi \right) (\xi) \right], \\ &= \frac{\partial^n}{\partial t^n} \left[t^{n-2\alpha-1} \left(n - 2\alpha - \frac{\alpha}{2} \xi \frac{\partial}{\partial \xi} \right) \left(\kappa_2^{\frac{1-\alpha, n-\alpha}{\delta}} \varphi \right) (\xi) \right], \\ &= t^{-2\alpha} \prod_{j=0}^{n-1} \left(1 - 2\alpha + j - \frac{\alpha}{2} \xi \frac{\partial}{\partial \xi} \right) \left(\kappa_2^{\frac{1-\alpha, n-\alpha}{\delta}} \varphi \right) (\xi), \\ &= t^{-2\alpha} \left(\rho_2^{\frac{1-2\alpha, n-\alpha}{\delta}} \varphi \right) (\xi).\end{aligned}$$

When $\alpha > 0$, the similarity transformation (20) transforms the time-fractional BT equation (1) into the subsequent subordinate non-linear ordinary fractional differential equation:

$$\left(\rho_2^{\frac{1-2\alpha, n-\alpha}{\delta}} \varphi \right) (\xi) = [K_1 \varphi(\zeta) + K_2 \zeta \dot{\varphi}(\zeta) + K_3 \zeta^2 \ddot{\varphi}(\zeta)],$$

where K_1, K_2, K_3 are constants depending on α .

5. The Time Fractional Bagley-Torvik (BT) Equation's Conservation Laws

As an isolated system evolves, the total quantity of a certain physical quantity stays constant, according to a fundamental principle of physics. This idea is called a conservation law in mathematics. Applications of conservation rules are extensive and span many scientific fields. One of the best applications of conservation principles is the development of unique solutions for integro-differential equations, partial differential equations, and related systems. In 1918, Noether introduced a groundbreaking theorem establishing a fundamental connection between symmetry groups and conservation laws.

Noether's theorem provides a systematic way to derive the corresponding conservation laws for a system governed by the Euler-Lagrange equations when a Lagrangian exists. For systems involving fractional derivatives, several studies have explored the derivation of conservation laws via Noether's theorem (see (Agrawal, 2002) and (Riewe, 1996)). Building on this, Lukashchuk used the notion of a formal Lagrangian and the new conservation law theorem put out by Ibragimov (Ibragimov, 2007) to deduce fractional extensions of Noether's operators and get conservation laws for fractional diffusion-type equations (Lukashchuk, 2015). The main goal of this section is to apply Ibragimov's approach to obtain the conservation laws for the TF-BT equation. For the fractional BT equation (1), a continuity equation is used to define a conservation law in this situation.

$$D_t(C^t) + D_x(C^x) \Big|_{\text{Eq.(1)}} = 0, \quad (25)$$

where the functions of t, x, u , and derivatives of u are the conserved vectors. The entire derivative of variable i is shown here by D_i . The formal Lagrangian of the fractional equation (1) is therefore as follows:

$$\mathcal{L} = V(t, x, u) F_j = V(a u_{tt} + b {}^C D_t^\alpha u + cu), \quad (26)$$

And the new dependent variable is denoted by $V(t, x, u)$. As a result, equation (1)'s adjoint equation has the following form.

$$F^* = \frac{\delta \mathcal{L}}{\delta u} = 0 \quad (27)$$

where $\frac{\delta \mathcal{L}}{\delta u}$ is the variable H 's Euler-Lagrange operator, which is defined as follows:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial ({}^C D_t^\alpha u)} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_{xx} \frac{\partial}{\partial u_{xx}} + D_{tt} \frac{\partial}{\partial u_{tt}} - \dots$$

where $(D_t^\alpha)^*$ is the operator that is adjoint to ${}^C D_t^\alpha u$. For the Fractional Caputo derivative, the adjoint operator is as

$$({}^C D_t^\alpha)^* = {}_t D_T^\alpha$$

Apply Ibragimov's Theorem. For a Lie point symmetry of equation (8):

$$C^t = W I_t^{1-\alpha} \frac{\partial \mathcal{L}}{\partial ({}^C D_t^\alpha u)} - J(D_t(W), \frac{\partial \mathcal{L}}{\partial ({}^C D_t^\alpha u)}) + W \frac{\partial \mathcal{L}}{\partial u_t} - D_x(W) \frac{\partial \mathcal{L}}{\partial u_{tt}} \quad (28)$$

$$C^x = \xi \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial u_x} - D_x(W) \frac{\partial \mathcal{L}}{\partial u_{xx}} + \dots \quad (29)$$

Where

$$W = \eta - \tau u_t - \xi u_x. \quad (30)$$

The integral J in (28) is defined by

$$J(f, g) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_t^T \frac{f(\tau, x) g(\mu, x)}{(\mu - \tau)^\alpha} d\mu d\tau.$$

Let's choose the symmetry generator:

$$V_1 = \partial_t \Rightarrow \tau = 1, \xi = 0, \eta = 0 \Rightarrow W = -u_t$$

Then, from equation (8), one can compute $\frac{\partial \mathcal{L}}{\partial u_t} = 0$, $\frac{\partial \mathcal{L}}{\partial u_{tt}} = av$, $D_t(W) = -u_{tt}$.

$$C^t = v[-u_t \Gamma(\alpha) - I(-u_t, (T-t)^{\alpha-1})] + av u_{tt},$$

$$C^x = 0.$$

Let's choose the symmetry generator:

$$V_2 = \partial_x \Rightarrow \tau = 0, \xi = 1, \eta = 0 \Rightarrow W = -u_x$$

Then, from equation (8), one can obtain $\frac{\partial \mathcal{L}}{\partial u_t} = 0$, $\frac{\partial \mathcal{L}}{\partial u_x} = 0$, $\frac{\partial \mathcal{L}}{\partial u_{tt}} = av$, $D_t(W) = -u_{xt}$

$$C^t = v[-u_x \Gamma(\alpha) - I(-u_x, (T-t)^{\alpha-1})] + av u_{xt},$$

$$C^x = \mathcal{L} = v(au_{tt} + b {}^C D_t^\alpha u + cu) = vF(u).$$

Let's choose the symmetry generator:

$$V_3 = t \partial_t + \frac{\alpha}{2} x \partial_x - \alpha u \partial_u \Rightarrow \tau = 1, \xi = \frac{\alpha}{2} x, \eta = -\alpha u,$$

$$W = \eta - \tau u_t - \xi u_x = -\alpha u - t u_t - \frac{\alpha}{2} x u_x.$$

Thus, from equation (8), one can results $\frac{\partial \mathcal{L}}{\partial u_t} = 0$, $\frac{\partial \mathcal{L}}{\partial u_x} = 0$, $\frac{\partial \mathcal{L}}{\partial u_{tt}} = av$.

$$D_t(W) = -\alpha u_t - u_t - t u_{tt} - \frac{\alpha}{2} x u_{xt} = -(\alpha + 1) u_t - t u_{tt} - \frac{\alpha}{2} x u_{xt}$$

$$C^t = v[W \Gamma(\alpha) - I(W, (T-t)^{\alpha-1})] + W \frac{\partial \mathcal{L}}{\partial u_t} - D_t(W) \frac{\partial \mathcal{L}}{\partial u_{tt}} = v[W \Gamma(\alpha) - I(W, (T-t)^{\alpha-1})] + av D_t(W),$$

$$C^x = 0.$$

Thus, we have

$$C^t = v\left[\left(-\alpha u - t u_t - \frac{\alpha}{2} x u_x\right) \Gamma(\alpha) - I\left(\left(-\alpha u - t u_t - \frac{\alpha}{2} x u_x\right), (T-t)^{\alpha-1}\right)\right] + av\left(t u_{tt} + (\alpha + 1) u_t + \frac{\alpha}{2} x u_{xt}\right) +$$

$$tv(b {}^C D_t^\alpha u + cu),$$

$$C^x = 0.$$

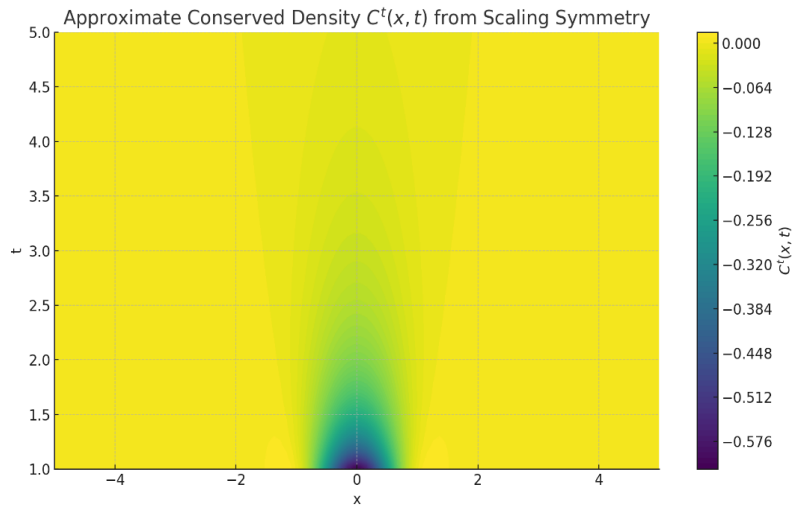


Figure 2: Here is the Plot of the Conserved Density $C^t(t, x)$ Associated with the Scaling Symmetry of the Time-Fractional Bagley–Torvik Equation.

6. Conclusions

The purpose of this study is to show how well the Lie symmetry approach works for assessing the TF-BT formula. For the fractional differential equation under consideration, conservation rules are constructed using the deduced Lie point symmetries. Despite the absence of a classical Lagrangian for RL derivatives of order $0 < \alpha < 1$, the methods outlined herein enable the formulation of conservation laws for such equations. The findings validate the successful expansion of Lie group analysis to a diverse type of FDEs, providing exact solutions and structural insights into these equations. It is anticipated that this will lead to new research in mathematical finance using Lie symmetry methods. Numerical methods for the FBTE will also be explored in future work.

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