

# Analysis of A Novel Fractional-Order Four-Dimensional Dynamical System with Sinusoidal Term for EEG Signal: A Stability and Hyperchaotic Perspective

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## ABSTRACT

This paper explores a novel fractional-order modification of the Rössler system via the Caputo differential operator. The dynamical system is augmented to a four-dimensions space by appending a four-state variable, thereby increasing the dynamical complexity of the new system and enabling it to produce hyperchaotic dynamics. A sinusoidal function is additionally included within the governing equations, giving rise to novel class of nonlinear dynamical interaction compared to the original Rossler system. The equilibrium states of the fractional-order system (1) are computed, and the stability behavior in the vicinity of these points is explored through the use of the linearization matrix coupled with the well-known Matignon's stability criterion. The system analytically shown to be dissipative, which establishes the fact that its trajectories are bounded with the evolution of time. The dynamical characteristic of system (1) is examined via computational simulations using FDE12 method, the Adams-Bash fourth-Moulton predictor-corrector algorithm. The obtained results confirm the existence of distinct dynamical regimes, including chaotic and hyperchaotic attractors. These results are affirmed by means of the Lyapunov exponents and strengthened through the 0–1 diagnostic test for chaos. The impact of varying the system parameters is also highlighted through bifurcation diagram. To apply system (1) to recorded real data, the governing parameters are determined via genetic algorithm based on EEG recordings, retrieved from the PhysioNet open-access database. Signals from the Fz, Cz, Pz channels and parietal midline Wz are used. The comparison between simulated and real data shows similar dynamical patterns, indicating that the model can capture important features of neural activity. Overall, the proposed system (1) provides a useful framework for studying complex nonlinear dynamics and may be applied in the analysis of biological signals.

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Hyperchaos;  
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## 1. Introduction

Nonlinear dynamical systems have been widely investigated due to their capability to describe complex phenomena observed in physics, engineering, biology, and neuroscience. Among the classical chaotic systems, the [Rössler \(1976\)](#) system represents one of the simplest continuous-time models capable of generating chaotic attractors with relatively simple nonlinear equations. Since its introduction, the Rössler system has become a fundamental model for studying chaotic behavior, bifurcation structures, and nonlinear dynamics in deterministic systems.

In recent decades, fractional calculus has emerged as an effective mathematical tool for modeling systems with memory and hereditary characteristics. Unlike classical integer-order derivatives, fractional derivatives allow the description of processes in which the present state relies on the whole past behavior of the dynamical system ([Caputo, 1967](#)). Because of this property, fractional-order models have been successfully applied in many scientific areas including viscoelastic materials, electrical circuits, biological systems, and nonlinear dynamical systems ([He et al., 2015](#)).

Among the formulations of fractional-order derivatives, the [Caputo \(1967\)](#) derivative is extensively employed in practical modeling because it allows the use of physical meaningful initial conditions expressed in terms of integer-order derivatives. The introduction of fractional calculus into chaotic systems has significantly expanded the study of nonlinear dynamics, as fractional-order systems often exhibit richer and more complex behaviors compared with their integer-order

counterparts (Li & Chen, 2004).

The stability analysis of fractional-order systems requires mathematical tools from those used in classical integer-order systems, in this context, Matignon (1996) developed a stability criterion that provides necessary conditions for the asymptotic stability of fractional-order systems. this criterion has become one of the fundamental tools for analysis the stability of fractional dynamical systems. Another important research in nonlinear dynamics is the development of modified chaotic systems capable of generating more complex behaviors. by introducing additional nonlinear terms or increasing dimensionality of the system. New dynamical structures can emerge, including hyperchaotic attractors characterized by the presence of multiple positive Lyapunov exponent (Ott, 2002). Hyperchaotic systems have attracted considerable interest due to their higher level of unpredictability and their potential applications in secure communications, signal processing, and complex system modeling (Ivanov et al., 2001).

Motivated by these considerations, the present work proposed developed fractional-order four-dimensional Rossler system formulated using the Caputo fractional derivative (Zhang et al., 2024). The new proposed system extends the classical Rossler structure by introducing an additional state variable, together with a sinusoidal term the incorporation of the nonlinear function,  $\sin(w)$  introduces a new interaction mechanism within the system and significantly enriches its dynamical behavior, the dynamical properties of system (1) are analyzed by mean of both theoretical and computational methods. The steady-state points are firstly derived, and the local and global dynamical stability of the fractional-order system is examined using the Jacobian matrix together with Matignon's stability criterion and Lyapunov function. furthermore, the dissipative of the system is analyzed to ensure that the system trajectories remain a bounded within a finite region of the phase plane (Elbadri et al., 2025).

To explore the nonlinear properties associated with the new system (1), numerical simulations are performed using the FED<sub>12</sub> numerical method for solving fractional differential equation (Diethelm et al., 2002). The results obtained reveal various dynamical regimes including periodic motion, chaotic attractors, and hyperchaotic behavior. In addition, bifurcation diagrams and Lyapunov exponent analysis are employed to provide deeper insight into the complex dynamics of the system.

Finally, the chaotic characteristics of the new system are further investigated using the

0–1 test for chaos. This test is applied to both the simulated signals generated from the new system and real electroencephalography (EEG) signals obtained from epilepsy patients. The comparison between simulated data and real physiological signals provides further evidence of the ability of the proposed model to reproduce complex dynamical behaviors observed in real-world systems.

Several criteria were considered when selecting and adding the sine function to a dynamic system, including:

1) Introducing oscillatory non-contradiction into the system, where the sine function generates natural oscillations that add periodic force to the system. This force can lead to oscillatory or chaotic activity similar to that found in biological systems, which are considered a model for explaining chaotic systems.

2) The sine function also causes the system to deviate from its equilibrium state and promotes the presence of a chaotic attractor or even a hyperchaotic system. The periodic oscillations resulting from the sine function in dynamic systems are extremely useful in applications such as monitoring biological systems, cryptography, engineering and physical applications, and system simulation, thus improving results and outputs.

## 2. Formulation of the Four-Dimensional Fractional-Order System

The four-dimensional fractional-order system formulated in (1), developed from Rossler system and derived by using the Caputo fractional derivatives, whose order  $\alpha \in (0,1]$ , which generalizes the classical system while incorporating memory effects. The following system of equations has one quadratic and one sinusoidal nonlinearity; terms represent the new system (Liu et al., 2009):

$$\left. \begin{aligned} {}^C D^\alpha X(t) &= -y(t) - z(t) \\ {}^C D^\alpha Y(t) &= x(t) + ay(t) \\ {}^C D^\alpha Z(t) &= b + z(t)(x(t) - c) \\ {}^C D^\alpha w(t) &= \beta x + \sin(w) \end{aligned} \right\} \quad (1)$$

Where  ${}^C D^\alpha$  denotes the Caputo fractional derivative of order  $(\alpha)$  and  $a, b, c$  and  $\beta$  are real parameters controlling the system behavior and  $0 < \alpha \leq 1$  is the fractional order (with  $\alpha = 1$  recovering the classical system).

### 2.1 Caputo Derivative Definition

Caputo derivative definition of order  $\alpha \in (0,1]$  as:  ${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau$  (2)

This form allows for initial conditions of integer order making it more suitable for physical and biological modeling

than Riemann-Liouville derivatives.

In this paper a Real data taken from electroencephalography (EEG) where ( $xt = Fz$ : frontal midline,  $yt = Cz$ : vertex,  $zt = Pz$ : parietal midline, and  $wt = Wz$ : it is an internal phase variable) (Ahmed et al., 2025).

The real signals were taken from <https://physionet.org/databases>, were used in the process of estimating the model parameters. A genetic algorithm was employed to select the optimal parameter set that achieves the highest agreement between the numerical system's behavior and the real data. This contributed to obtaining an accurate and reliable representation of the dynamics extracted from the biological signals. with the fractional derivation degree  $\alpha = 0.95$  and Parameters: ( $a = b = 0.201$ ,  $c = 6.5$ ,  $\beta = 0.35$ ).

## 2.2 Numerical and Graphical Analysis

### 2.2.1 Fractional Differential Equation Numerical Method (FDE12) (Brandibur et al., 2021; Garrappa, 2010)

It is a numerical algorithm for solving fractional differential equations. and first developed by the Italian researcher (Roberto Garrappa) in 2008 as a routine procedure for the MATLAB program. It is a method taken from the numerical method (Adam-Bash fourth Moulton) that was further developed in 2010 to be improved in terms of accuracy and depends on the (predictor-corrector) formula. It is also developed specifically to solve fractional differential equations derived in the Caputo derivative formula only because it is the most widely used in physical, engineering and biological applications.

#### 2.2.1.1 Description Method (Brandibur et al., 2021; Garrappa, 2010)

The time with period  $[0, T]$  can be divided into  $N$  steps. Where  $h = \frac{T}{n}$ ,  $nh = T$

The main parameters used in the estimation and corrective part can be defined as follows:

$$a_j = (j+1)^{\alpha+1} - 2j^{\alpha+1} + (j-1)^{\alpha+1}, (j \geq 1, a_0 = 1) \quad \& \quad b_j = (j+1)^{\alpha} - j^{\alpha}, j \geq 0$$

#### 2.2.1.2 The Predictor Formula of the Method

$$y_{n+1}^p = y_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^n b_{n-j} f(t_j, y_j) .$$

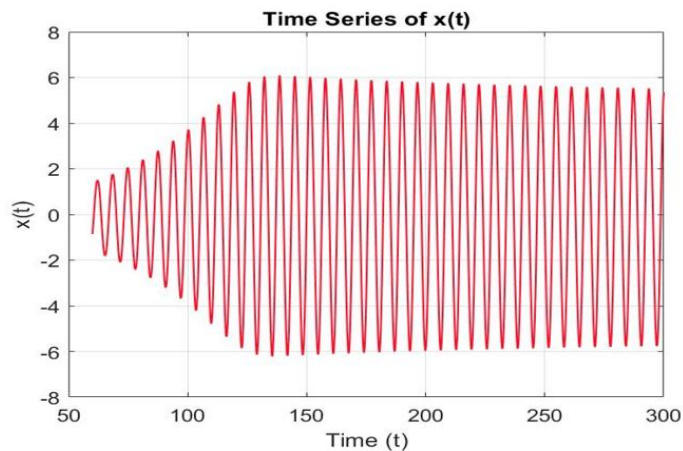
#### 2.2.1.3 The Corrector Formula of Method

$$y_{n+1} = y_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} [ f(t_{n+1}, y_{n+1}^p) + \sum_{j=0}^n a_{n-j} f(t_j, y_j) ] , \text{ where } 0 < \alpha \leq 1$$

The advantages of this improved method over the numerical method are accuracy and stability, as it is good for solid problems if ( $h$ ) is small. Among the applications used in it, which are very commonly used, is in the analysis of fractional systems such as the fractional Rossler system and other fractional dynamic systems. a figure (1) shown the state space ( $x$  vs  $t$ ), ( $y$  vs  $t$ ), ( $z$  vs  $t$ ) and ( $w$  vs  $t$ ) of fractional-order system (1).

### 2.2.2 Graphical Analysis

By fixed  $\alpha = 0.95$  and parameters ( $a=b=0.201$ ,  $c=6.5$  and  $\beta=0.35$ ). In Figure (1) show the phase space of variables over time, where the variable  $x$  representing the frontal midline in the EEG, the variable  $y$  representing the vertex, the variable  $z$  representing the preintial midline and a variable ( $w$ ) representing the internal phase variable.



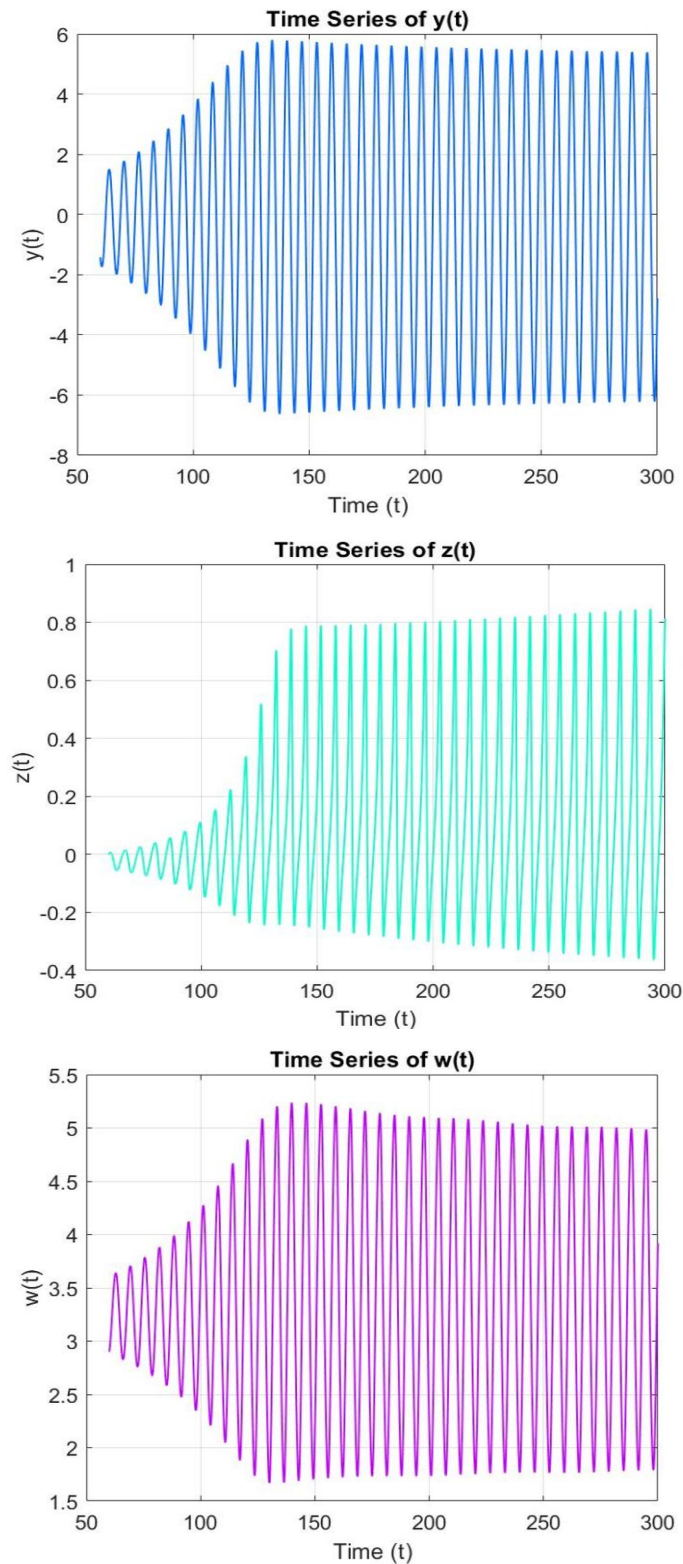


**Figure 1:** State Space:  $(x \text{ vs } t)$ ,  $(y \text{ vs } t)$ ,  $(z \text{ vs } t)$  and  $(w \text{ vs } t)$  of Fractional-Order System (1)

In figure (2) shows the two-dimensional phase-space diagrams of the developed fractional quadruple system, through the projections  $(x, y)$ ,  $(x, z)$ ,  $(y, z)$ ,  $(x, w)$ ,  $(y, w)$ , and  $(z, w)$ . Traditional projections reveal the basic dynamical behavior, while projections incorporating the variable  $w$  reveal the influence of sinusoidal nonlinearity  $\sin(w)$  in shaping the attractor's structure.

The results demonstrate complex dynamical behavior characterized by overlapping and ordered orbits, reflecting the

role of 4 – D expansion and sinusoidal nonlinearity in enhancing nonlinear interactions and enriching the system's properties compared to the classical model.

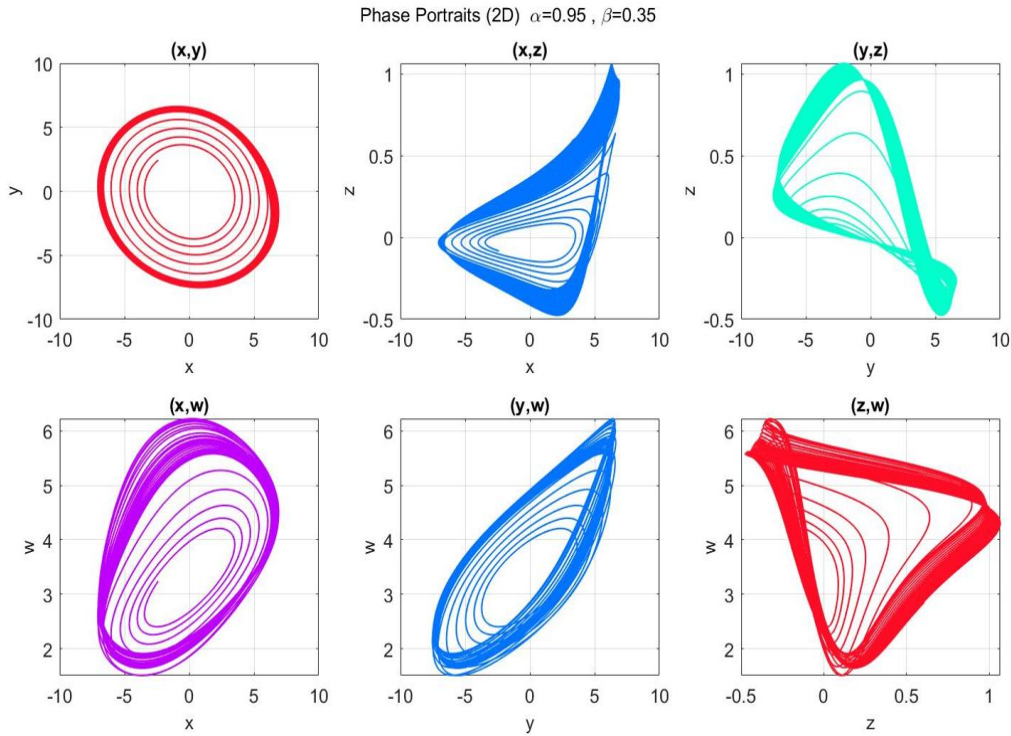


**Figure 2:** Phase Diagram of System (1) With Fractional-Order at ( $\alpha=0.95$ ) and ( $a = b = 0.201, c = 6.5, \beta = 0.35$ ) and Initial Condition (0.1,0.1,0.1,0.1), ( $x$  vs  $y$ ), ( $x$  vs  $z$ ), ( $y$  vs  $z$ ), ( $x$  vs  $w$ ), ( $y$  vs  $w$ ) and ( $z$  vs  $w$ ) Attractor of ( $x, y, z, w$ )

## 2.3 Equilibrium Points

To find the equilibrium points of system (1), we solve by equal the equations to zero ( ${}^cD^\alpha x = {}^cD^\alpha y = {}^cD^\alpha z = {}^cD^\alpha w = 0$ ) and obtain from the first equation of the system ( $z = -y$ ), and from the second equation ( $x = -ay, x = az, y = -z$ ), substituting into third equation to obtaining the quadratic equation as the following:

$$[b + (-y)(-ay - c)] = 0 \rightarrow b + y(ay + c) = 0 \rightarrow ay^2 + cy + b = 0 \quad (3)$$

Solving this quadratic equation in ( $y$ ):  $ay^2 + cy + b = 0$ , The fourth equation in system (1) which contains the sine function and the fourth variable:  $\beta x + \sin(w) = 0 \rightarrow \sin(w) = -\beta x \rightarrow \sin(w) = -\beta(az)$ . This gives one or two equilibrium points depending on the discriminant:

$$\Delta = C^2 - 4ab, \text{ (where } a, b, c > 0 \text{ and } a, b < c)$$

Then the equilibrium points of the fractional- order system is:

$E_0: (0,0,0,0) = E_{1,2} \left( \frac{c \mp \sqrt{c^2 - 4ab}}{2}, \frac{-c \mp \sqrt{c^2 - 4ab}}{2a}, \frac{c \mp \sqrt{c^2 - 4ab}}{2a}, w \right)$ . From the fourth equation in system (1) is:  $\beta x + \sin(w) = 0 \rightarrow \sin(w) = -\beta x$ , for a real solution to exist the following condition must be met:  $|\beta x| \leq 1$ , because of the periodicity of sin, we obtain an infinite number of solutions to  $w^*$  where:

$$w = (-1)^k \arcsin(-\beta x) + k\pi, \quad k \in \mathbb{Z}.$$

Two real equilibrium points provided that:  $C^2 - 4ab > 0$

### 2.3.1 Proposition (1): (Equilibrium Points)

The equilibrium points of the fourth-dimensional-order system (1) are given by:

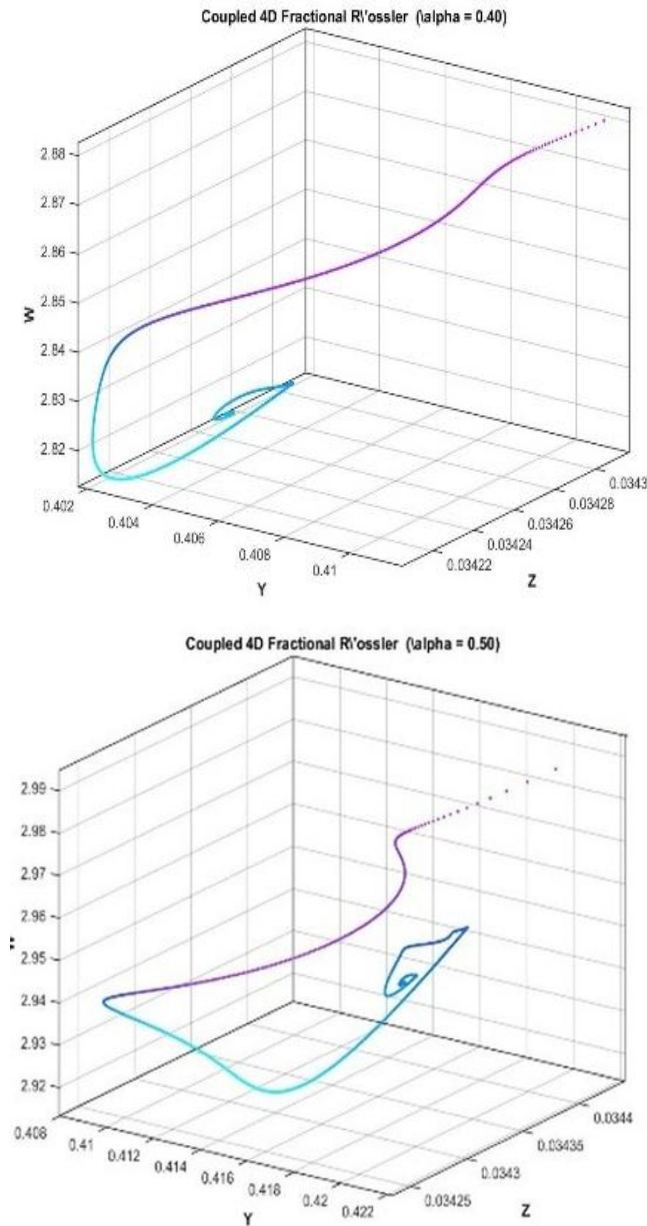
$$E_0(0,0,0,0), \quad E_{1,2} \left( \frac{c \mp \sqrt{c^2 - 4ab}}{2}, \frac{-c \mp \sqrt{c^2 - 4ab}}{2a}, \frac{c \mp \sqrt{c^2 - 4ab}}{2a}, w \right),$$

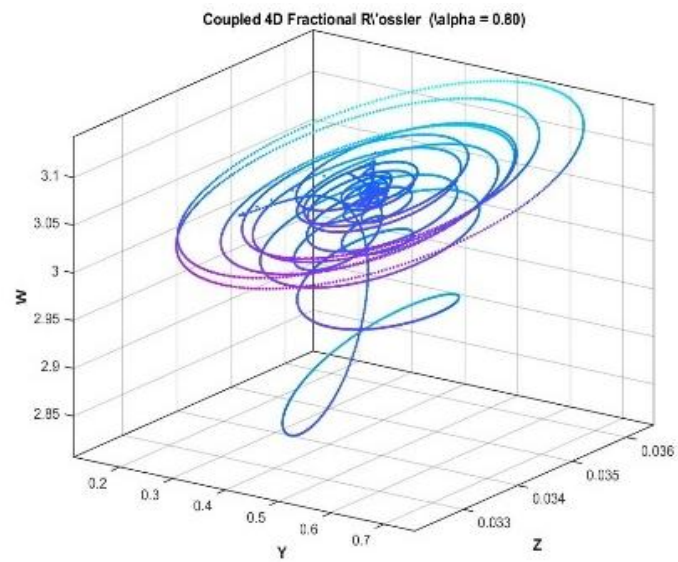
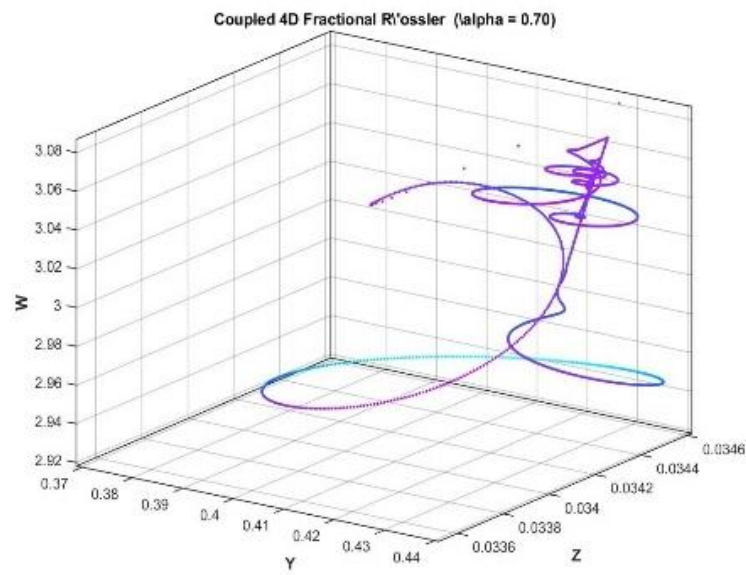
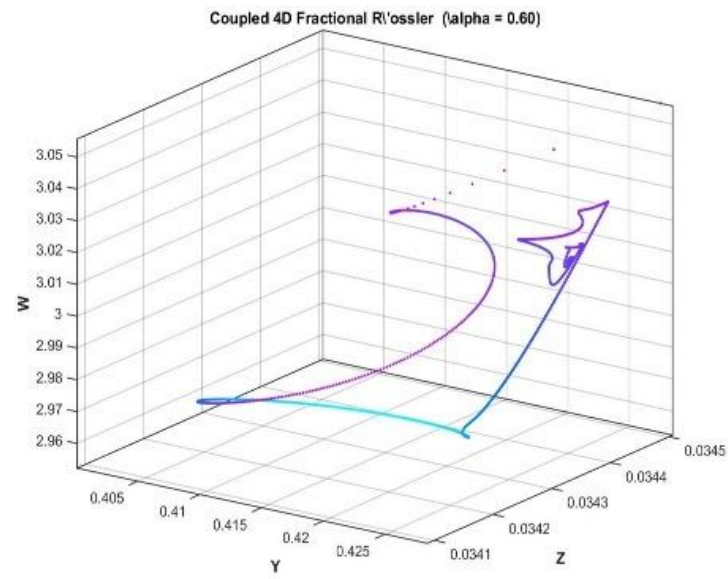
If  $\Delta = C^2 - 4ab > 0$ , and  $|\beta x^*| \leq 1$  then the equilibrium points are given by:  $E^k = (x, y, z, w)_k$ , where  $z = \frac{c \pm \sqrt{c^2 - 4ab}}{2a}$ ,  $y = -z$  and  $x = az$  and  $w^{(k)} = (-1)^k \arcsin(-\beta x) + k\pi$ ,  $k \in \mathbb{Z}$  and if  $\Delta < 0$  then no real equilibrium exists in the  $(x, y, z, w)$ -subsystem.

Proof: The proof follows directly, by equating the right-hand side of System (1) to zero and solving the resulting algebraic equations sequentially for  $(x, y, z, w)$  as shown above.

## 2.4 Attractor Visualization

The evolution of the three-dimensional attractor  $(y, z, w)$  of the tetragonal system developed at different fractional orders  $\alpha$ . The comparison shows how fractional “memory” affects the attractor’s structure, density, and transition between dynamic modes (Mirasso et al., 1997). A numerical predictor-corrector solution (such as FDE<sub>12</sub>) was adopted with the transient phase omitted to ensure accurate representation of the long-term behavior. Figure (3) shows the attractor phases for the fractional system with degrees of differentiation ( $\alpha = 0.4, 0.5, 0.6, 0.7, 0.8$  and  $0.95$ ).





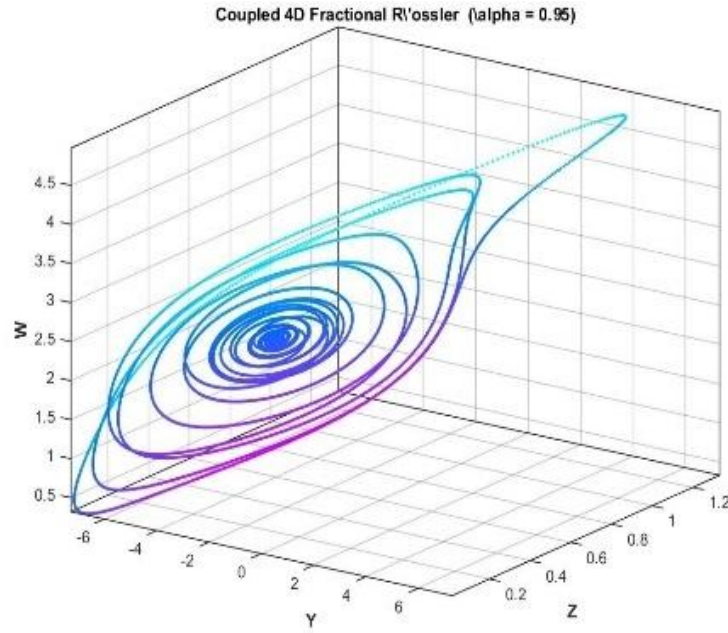


**Figure 3:** Shows the Attractor Phases for the Fractional System with Degrees of Differentiation  $a(\alpha=0.4)$ ,  $b(\alpha=0.5)$ ,  $c(\alpha=0.6)$ ,  $d(\alpha=0.7)$ ,  $e(\alpha=0.8)$ ,  $f(\alpha=0.95)$ .

### 3. Stability Analysis of the Fractional Order System

#### 3.1 The Jacobina Matrix

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial w} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} & \frac{\partial \dot{y}}{\partial w} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial w} \\ \frac{\partial \dot{w}}{\partial x} & \frac{\partial \dot{w}}{\partial y} & \frac{\partial \dot{w}}{\partial z} & \frac{\partial \dot{w}}{\partial w} \end{bmatrix} \rightarrow J = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a & 0 & 0 \\ z & 0 & x-c & 0 \\ \beta & 0 & 0 & \cos(w) \end{bmatrix}$$

Characteristic equation of system (1) from the following:

$$\begin{aligned} & |\lambda^\alpha I - J| = 0 \\ & = \left| \begin{bmatrix} \lambda^\alpha & 0 & 0 & 0 \\ 0 & \lambda^\alpha & 0 & 0 \\ 0 & 0 & \lambda^\alpha & 0 \\ 0 & 0 & 0 & \lambda^\alpha \end{bmatrix} - \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a & 0 & 0 \\ z & 0 & x-c & 0 \\ \beta & 0 & 0 & \cos(w) \end{bmatrix} \right| = 0 \\ \Rightarrow & = \left| \begin{bmatrix} \lambda^\alpha & 1 & 1 & 0 \\ -1 & \lambda^\alpha - a & 0 & 0 \\ -z & 0 & \lambda^\alpha - (x-c) & 0 \\ -\beta & 0 & 0 & \lambda^\alpha - \cos(w) \end{bmatrix} \right| = 0 \end{aligned}$$

The determinant of the matrix is calculated based on the first row (using Laplace's first-row method) and is as follows:

$$\det(\Delta) \equiv a_{11}, M_{11} - a_{12}, M_{12} + a_{13}, M_{13} - a_{14}, M_{14}, \quad a_{14} = 0 \quad \therefore a_{14}, M_{14} = 0$$

$$\det(\Delta) \equiv a_{11}, M_{11} - a_{12}, M_{12} + a_{13}, M_{13}$$

$$\det(\Delta) = \lambda^\alpha \cdot \begin{vmatrix} \lambda^\alpha & 0 & 0 \\ 0 & \lambda^\alpha - (x-c) & 0 \\ 0 & 0 & \lambda^\alpha - \cos(w) \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 0 & 0 \\ -z & \lambda^\alpha - (x-c) & 0 \\ -\beta & 0 & \lambda^\alpha - \cos(w) \end{vmatrix} +$$

$$1. \begin{vmatrix} -1 & \lambda^\alpha - a & 0 \\ -z & 0 & 0 \\ -\beta & 0 & \lambda^\alpha - \cos(w) \end{vmatrix} = 0$$

$$\lambda^\alpha [(\lambda^\alpha - a)(\lambda^\alpha - (x - c))(\lambda^\alpha - \cos(w))] + [(\lambda^\alpha - (x - c))(\lambda^\alpha - \cos(w))] + [(\lambda^\alpha - \cos(w))(\lambda^\alpha - a)(-z)] = 0$$

$$(\lambda^\alpha - \cos(w))[\lambda^\alpha(\lambda^\alpha - a)(\lambda^\alpha - (x - c)) + (\lambda^\alpha - (x - c)) + z(\lambda^\alpha - a)] = 0 \quad (4)$$

Either:  $\lambda^\alpha - \cos(w) = 0 \Rightarrow \lambda_1^\alpha = \cos(w)$ , The first real root of equation (4)

$$Or: [\lambda^\alpha(\lambda^\alpha - a)(\lambda^\alpha - (x - c))] + (\lambda^\alpha - (x - c)) + z(\lambda^\alpha - a) = 0 \quad (5)$$

Expansion polynomial: Let  $L = \lambda^\alpha$ , for easy substitution in Eq. (5)

$$[L(L - a)(L - (x - c)) + (L - (x - c)) + z(L - a)] = 0$$

$$P(L) = L^3 - (a + (x - c))L^2 + a(x - c)L + (L - (x - c)) + z(L - a) = 0$$

$$P(L) = L^3 - (a + (x - c))L^2 + (a(x - c) + z + 1)L - (az + (x - c)) = 0 \quad (6)$$

Cubic equation: To solve a cubic equation, factor its roots, and determine their type, the equation must be solved. Therefore, the roots of the equation (4) are:  $\{\cos(w), L_1, L_2, L_3\}$

We can transform the cubic equation (6) into a reduced cubic equation for the term  $(L^2)$  using Cardano's method, resulting in the form  $(L^3 + PL + q) = 0$ . Solve it and find the values of the roots.

let  $L = \mu - \frac{A}{3}$  ..... (7) then substitute eq. (7) in eq. (6), We are getting the reduced cubic equation:

$$\mu^3 + \underbrace{\left(B - \frac{A^2}{3}\right)}_p \mu + \underbrace{\left(\frac{2A^3}{27} - \frac{AB}{3} + C\right)}_q = 0 \quad (8)$$

By solving it, we get the roots of the reduced cubic equation:

$$\mu_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}, \quad \mu_2 = w \mu_1, \quad \mu_3 = w^2 \mu_1$$

Where:  $w = \frac{-1}{2} + i \frac{\sqrt{3}}{2}$  (Complex unit root)

$$\text{If } (\lambda^\alpha) = \mu_k - \frac{A}{3}, \Psi = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 \text{ then: } \mu_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{\Psi}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Psi}}$$

$$\lambda^{\alpha_1} = \sqrt[3]{\frac{-q}{2} + \sqrt{\Psi}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Psi}} - \frac{A}{3}, \quad \lambda^{\alpha_2} = w \sqrt[3]{\frac{-q}{2} + \sqrt{\Psi}} + w \sqrt[3]{\frac{-q}{2} - \sqrt{\Psi}} - \frac{A}{3}$$

$$\lambda^{\alpha_3} = w^2 \sqrt[3]{\frac{-q}{2} + \sqrt{\Psi}} + w^2 \sqrt[3]{\frac{-q}{2} - \sqrt{\Psi}} - \frac{A}{3}.$$

Here the values of the roots may be real or complex depending on the value of  $(\Psi)$ .

To find equilibrium points we need the parameters value of system (1), it can be estimated by using genetic algorithm and getting on:  $(a = b = 0.201, c = 6.5, \beta = 0.35)$  and  $(\alpha = 0.95)$ .

Then  $\Delta = c^2 - 4ab \rightarrow \Delta = (6.5)^2 - 4(0.201)(0.201) \rightarrow \Delta \approx 42.09$

$Z_{1,2} = \frac{6.5 \mp \sqrt{42.09}}{0.402}$ :  $Z_1 \approx 32.307$ , or  $Z_2 \approx 0.03065$  substitute and find  $x, y$  values  $x_1 \approx 6.4935, x_2 \approx 0.0065, y_1 = -32.307, y_2 = -0.03065$ .

We find  $w$  values as:  $\sin(w_1) = -(0.35)(6.4935) \approx -2.27 \rightarrow |-2.27| = 2.27 > 1$

This is impossible because it does not meet the condition  $|\sin(w_1)|$ , therefore the first equilibrium is not real.  $(\sin(w_1) \in [-1, 1])$   $\sin(w_2) = -(0.35)(0.0065) \approx -0.002275 \rightarrow |-0.002275| = 0.002275 < 1$  is acceptable.

$$W_{1,k} = -\arcsin(w_2) + 2k\pi$$

$E_0: (x_0, y_0, z_0, w_0) = (0, 0, 0, 0)$ ,  $E_{1,k} = (0.0065, -0.0306, 0.0306, -0.002275 + 2k\pi)$ , and  $E_{2,k} = (0.0065, -0.0306, 0.0306, 3.1438 + 2k\pi)$ , the system possessed a countably infinite equilibrium points which fall into

three categories the trivial equilibrium  $E_0$  and two parametric families  $E_{1,k}$  and  $E_{2,k}$  owing to this infinite equilibrium structure, the system emerges as a natural for supporting hidden attractors in the sense introduced by Leonov Kuznetsov.

Roots of the first family:  $\lambda^{\alpha}_1 = -6.453235$ ,  $\lambda^{\alpha}_{2,3} = 0.080228 \mp i 0.998891$  (two conjugate roots),  $\lambda^{\alpha}_4 = \cos(w_2, k) = 0.999998$ . Roots of the second family:  $\lambda^{\alpha}_1 = -6.453235$ ,  $\lambda^{\alpha}_{2,3} = 0.080228 \mp i 0.998891$  (two conjugate roots),  $\lambda^{\alpha}_4 = -0.999998$ . Calculating the eigenvalue angles (root angles for the Matignon criterion test:

$$1) |\arg(\lambda^{\alpha}_1)| = |\arg(-6.453235)| = \pi$$

$$2) \text{ Two conjugate complex roots: } \theta = |\tan^{-1}(\frac{\text{im}(\lambda^{\alpha})}{\text{re}(\lambda^{\alpha})})| = |\tan^{-1}(\frac{0.998891}{0.080228})| = |\mp 1.4906| = 1.4906$$

$$3) |\arg(\lambda^{\alpha}_4)| = |\arg(0.999998)| = 0, \text{ and } |\arg(\lambda^{\alpha}_4)| = |\arg(-0.999998)| = \pi$$

Then the angle of the first family roots is:  $(\pi, 1.4906, 1.4906, 0)$  is unstable for all  $0 < \alpha < 1$ .

Then the angle of the second family roots is:  $(\pi, 1.4906, 1.4906, \pi)$ .

### 3.2 The Stability Condition for Fractional System (Matignon Condition) (Brandibur et al., 2021)

A fractional-order system is locally asymptotically stable if and only if all eigenvalues  $(\lambda_i)$  of the Jacobian satisfy the condition:  $|\arg(\lambda_i)| > \alpha\pi/2$ , This is known as the Matignon stability criterion.

Theorem (1): [Locally stability under the fractional Matignon criterion]

Let  $E^* = (x, y, z, w)$  be an equilibrium point of the fractional-order system of order  $\alpha \in (0, 1]$ , consider the characteristic equation associated with the Jacobi matrix  $(J)$  at the equilibrium point  $E^*$  be defined by the relation:

$$|\lambda \alpha I - J| = \Phi(\lambda) \cos(w) = 0 \quad (9)$$

Where  $\Phi(\lambda)$  denoted to the cubic factor associated with the  $(x, y, z)$ - sub statement.

Then the equilibrium  $E^*$  is locally asymptotically stable in the fractional sense if and only if all eigenvalues  $(\lambda_i)$  of the characteristic equation of system satisfy the Matignon condition:

$$|\arg(\lambda^{\alpha} i)| > \frac{\alpha\pi}{2}$$

Proof: the characteristic equation separates naturally into two components:

1) The eigenvalues generated by the equation:

$$\cos(w) = \lambda^{\alpha}$$

2) The three eigenvalues determined by the roots of the cubic polynomial

$$\Phi(\lambda_i) = 0$$

The stability implication of the fourth dimension.

Since  $\cos(w) \in [-1, 1]$  is real

Then there are three distinct cases that arise:

❖ If  $\cos(w) > 0$  then the principal eigenvalue satisfies:  $\arg(\lambda) = 0$  which violates the fractional stability condition for any  $\alpha > 0$

❖ Hence the equilibrium is necessarily unstable.

❖ If  $\cos(w) = 0$  then  $\lambda = 0$ , this is considered a borderline equilibrium and is often decidedly unstable.

❖ If  $\cos(w) < 0$  the corresponding eigenvalue lies on the negative real axis, therefore:  $|\arg(\lambda)| = \pi$ , and since  $\frac{\alpha\pi}{2} < \pi$  for all  $0 < \alpha \leq 1$  this eigenvalue does not destroy stability accordingly the fourth dimension induces instability only when the selected branch of  $w^*$  yields a positive cosine value, asymptotically stable if and only if:

$$1) \cos(w) < 0 \rightarrow |\arg(\lambda)| = \pi$$

$$2) \text{ All eigenvalues obtained from the cubic factor satisfy: } |\arg(\lambda_i)| > \frac{\alpha\pi}{2}$$

Otherwise: the equilibrium is unstable.

### 3.3 Lyapunov Stability

The basic idea behind the Lyapunov function is to determine whether, starting near the equilibrium point  $E^*$ , the system moves closer to or further away from it. The Lyapunov function represents energy and is positively defined. and It measures the distance from the origin, not just from  $E^*$ . Therefore, in fractional dynamical systems, the sign of the results  $({}^c D^{\alpha} V)$  is not the only factor to consider. A displacement method is used to further confirm the stability of the dynamical system.

$$V(x, y, z, w) = \frac{1}{2}(x^2, y^2, z^2, w^2) \rightarrow {}^c D^\alpha V = x'X + y'Y + z'Z + w'W$$

$${}^c D^\alpha V = x(-y - z) + y(x + ay) + z(b + z(x - c)) + w(\beta x - \sin(w))$$

$${}^c D^\alpha V = -xy - xz + xy + ay^2 + bz + z^2(x - c) + w\beta x + w \sin(w)$$

By substitute  $x = az$ , we get:

$${}^c D^\alpha V = -(az)z + ay^2 + bz + z^2(az - c) + \alpha xw + w \sin(w)$$

$${}^c D^\alpha V = az^3 - az^2 - cz^2 + a(-z)^2 + bz + \alpha xw + w \sin(w) \text{ By substitute } y = -z, \text{ we get:}$$

$$: {}^c D^\alpha V = az^3 - cz^2 + bz + \alpha azw + w \sin(w) \dots\dots\dots (10)$$

Substituting the values of the parameters and variables into the equation (10) yields the value of the Lyapunov function. If  ${}^c D^\alpha V > 0$ , the system is unstable; if  ${}^c D^\alpha V = 0$ , the system is neutral; and if  ${}^c D^\alpha V < 0$ , the system is stable.

Stability of a 4-D fractional-order system using the displacement method. This is a standard method in Lyapunov analysis for testing the stability of a dynamical system when the derivative is a fraction, and it is as follows for the system.

$$e_x = x - x^*, e_y = y - y^*, e_z = z - z^*, e_w = w - w^*$$

These variables ( $e_x, e_y, e_z, e_w$ ) are called displacement variables about the equilibrium point; that is, if  $E = (x, y, z, w)$  is the equilibrium point, then the magnitude of the variable is about the equilibrium value.

By following this procedure, we determine the global stability of system (1), and whether the solutions are approaching the origin or moving away from it (and the origin is the point of stability of the system).

$$\therefore V(e) = \frac{1}{2} (e_x^2 + e_y^2 + e_z^2 + e_w^2) * P$$

The larger the value of the displacement variable ( $e$ ), the larger the value of ( $V_e$ ); therefore, it is called a positive definition.

Proposition (2): (limitation of diagonal quadratic Lyapunov construction) Consider the system of displacements around the equilibrium point ( $E$ ) for the improved quadrilateral system where ( $0 < \alpha \leq 1$ ) and assume that  $p$  is the quadratic Lyapunov candidate, which takes the form:

$$V(e) = \frac{1}{2} (P e_x^2 + P e_y^2 + P e_z^2 + P e_w^2)$$

Therefore, the fractional derivative cannot be made ( ${}^c D^\alpha V$ ) negative over the entire state space of the system using this candidate for any choice of positive constants ( $P$ ). In other words, the quadratic form does not provide evidence for the global stability of the equilibrium state.

Proof: let  $E = (x, y, z, w)$  be an equilibrium point of the new four-dimensional extended Rossler system (1) and define the shifted variables as:  $e_x = x - x^*, e_y = y - y^*, e_z = z - z^*$  and  $e_w = w - w^*$

Consider the quadratic Lyapunov function as:  $V(e) = \frac{1}{2} (P e_x^2 + P e_y^2 + P e_z^2 + P e_w^2)$  where  $p > 0$ , this function is positive define as that:

$$V(e) = \frac{1}{2} \sum p e_i^2, P > 0, \text{ for all } i \text{ } i = x, y, z, w$$

Using the fractional Lyapunov inequality:  ${}^c D^\alpha V e_i \geq {}^c D^\alpha (\frac{1}{2} e_i^2)$

However: after substituting the shifted system into the Caputo fractional derivative of  $V$ , the nonlinear term  $z(x - c)$  in the third state equation in system (1) produced the term of the form:

$P e_x e_z^2$ , for  $e_x > 0$  and sufficiently large  $e_z$ , this term becomes positive and many dominate the remaining negative term, hence no positive diagonal choice of ( $P$ ) can guarantee:

${}^c D^\alpha V(e) \leq 0$  globally over the whole state space.

Therefore, the standard diagonal quadratic Lyapunov function cannot establish global asymptotic stability of the equilibrium.

This result is consistent with the intrinsically complex dynamics of the extended system (1) structure

The proof is therefore complete when we show the existence of a trend in state space that makes  $D^\alpha V > 0$  whatever we choose the parameters of  $0 < P$ , and this is logically sufficient to invalidate global stability with the arbitrary ( $P$ ).

## 4. Dissipation Calculation for a Fractional-Order System (1)

The dissipative nature of the proposed four-dimensional fractional system can be examined through the divergence of its vector field. For the present model, the divergence is given by

$$\nabla \cdot f = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} \rightarrow \nabla \cdot f = a + (x - c) + \cos(w) \quad (11)$$

$$\nabla \cdot f = \sum_{i=1}^4 \text{Trace}(J), J = M4 \times 4, (\nabla \cdot f < 0 \text{ dissipative condition})$$

If the value of divergence ( $\nabla \cdot f$ ) negative this means that the system is Dissipative, and if the value of ( $\nabla \cdot f$ ) = 0 then the system is Conservative, and if the value of ( $\nabla \cdot f$ ) is positive then the system is unbounded.

which depends explicitly on the system states. This indicates a net contraction of phase-space volume, meaning that the flow compresses trajectories into a bounded invariant set. Such contraction is consistent with the numerical observation that the sum of Lyapunov exponents is negative, thereby confirming that the system is globally dissipative despite exhibiting multiple directions of local instability. Substitute the values of parameters in eq. (11) and testing the result  $\nabla \cdot f = (0.201) + (0.0065 - 6.5) + 0.999998 = -5.290002 < 0$  then the system is dissipative.

### 4.1 Bifurcation

By fixed  $\alpha = 0.95$  and parameters ( $a = b = 0.201$ ,  $\beta = 0.35$  and  $c$  from 2 to 10) and initial condition (0.1, 0.1, 0.1, 0.1) with step size  $h = 0.01$ , The following bifurcation diagram illustrates the dynamical evolution of the proposed fractional-order 4D system (1) as the control parameter ( $c$ ) varies, where the local maxima of the state variable are extracted after eliminating transient effects. The figure (4) exhibit a structured transition from regular periodic motion to increasingly complex dynamics, highlighting the critical role of nonlinear coupling and fractional memory in shaping the system's behavior.

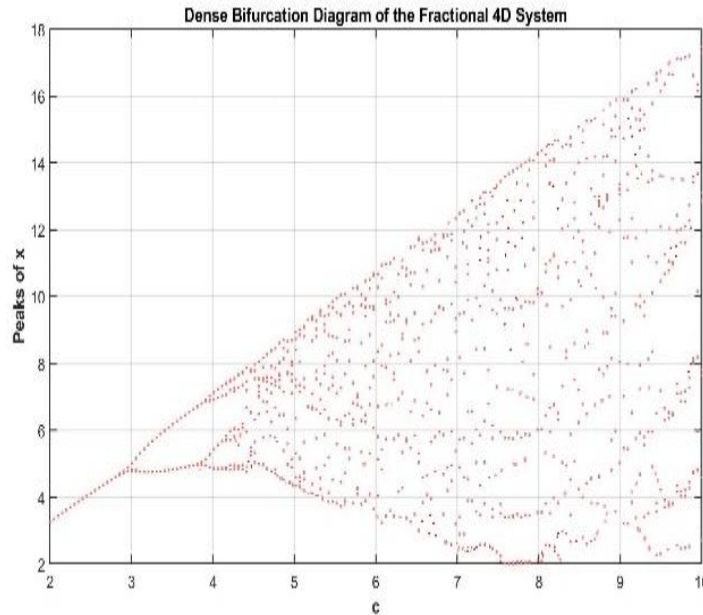


**Figure 4:** Show the Bifurcation Behavior  $c \in [2, 10]$  for 4-D Fractional-Order System (1).

## 5. Lyapunov Chaotic Analysis

### 5.1 Lyapunov Exponents (Danca, 2021; Li et al., 2023)

To rigorously characterize the dynamical behavior of the proposed four-dimensional fractional system, the complete Lyapunov spectrum was computed rather than relying solely on the largest Lyapunov exponent. While the largest exponent is sufficient to confirm the presence of chaos, it does not provide information about the dimensional structure of instability. Since hyperchaotic dynamics require the coexistence of at least two positive Lyapunov exponents, and thus evaluating the full spectrum becomes essential.  $L_1 = 0.026603$ ,  $L_2 = 0.016444$ ,  $L_3 = -0.480637$ ,  $L_4 = -5.566781$ . figure (5) shows the convergence of 4-D fractional-order new system.

The presence of two positive Lyapunov exponents confirms that the system operates in a hyperchaotic regime,

characterized by multidirectional exponential divergence in phase space.

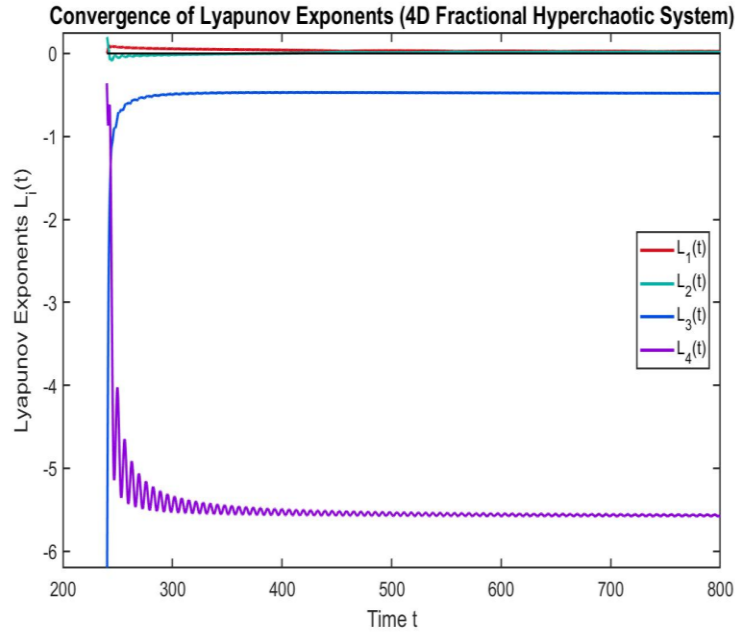


**Figure 5:** Lyapunov Exponents (LE1, LE2, LE3, LE4) of the Fractional-Order 4D Developed System (1)

## 5.2 (Kaplan–Yorke Dimension) ( $D_{KY}$ )

If we have the values of Lyapunov exponents  $LE_1, LE_2, LE_3, LE_4$  in 4-D dynamical system then the Lyapunov exponents are arranged in descending order:  $LE_4 \leq LE_3 \leq LE_2 \leq LE_1$ , then the searching is for largest integer sum. of the exponents such that:  $\sum_{i=1}^j LE_i > 0$  but  $\sum_{i=1}^{j+1} LE_i < 0$

Then:  $\nabla \cdot f \cong \sum_{i=1}^n LE_i < 0, n = 1, 2, \dots, j+1$  (system (1) is dissipative).

Then:  $D_{KY} = j + \frac{\sum_{i=1}^j LE_i}{|LE_{j+1}|} \rightarrow D_{KY} = 2 + \frac{LE_1 + LE_2}{|LE_3|} = 3 + \frac{0.026603 + 0.016444}{|-5.566781|} \approx 2.0896 > 0$

This fractional dimension reflects the geometrical complexity of the hyperchaotic attractor and provides a quantitative measure of its fractal structure.

## 6. Zero – One Test (Gottwald & Melbourne, 2009)

The binary test (0–1) for chaos is a simple and effective numerical tool that helps to distinguish between (regular periodic or quasi-periodic) dynamics and chaotic dynamics, including the Fractional-order dynamic system, the method is applied immediately to the temporal data, thereby eliminating the need for any reconstruction of the phase space;

Step (1): Compute Translation Variables: Given a time series  $\phi(j)$ , compute the translation variables  $p(n)$  and  $q(n)$  using  $P(n) = \sum_{j=1}^n \phi(j) \cos(jc)$ , and  $q(n) = \sum_{j=1}^n \phi(j) \sin(jc)$ , where  $c$  is a randomly chosen constant in  $(0, \pi)$ ,  $n = 1, 2, \dots, L$ .

Step (2): Compute Mean Square Displacement (MSD): calculate  $M(n)$  the Mean Square Displacement of the  $(p, q)$  trajectory.  $M(n) = \lim_{L \rightarrow \infty} \left( \frac{1}{L} \sum_{j=1}^L [(P(j+n) - P(j))^2 + (q(j+n) - q(j))^2] \right)$  Where  $n = 1, 2, \dots, L/10$ .

$$Vosc(n) = [E(\phi)]^2 \times \frac{1 - \cos(nc)}{1 - \cos(c)}, \text{ where } E(\phi) = \lim_{L \rightarrow \infty} \left( \frac{1}{L} \right) \sum_{j=1}^n \phi(j)$$

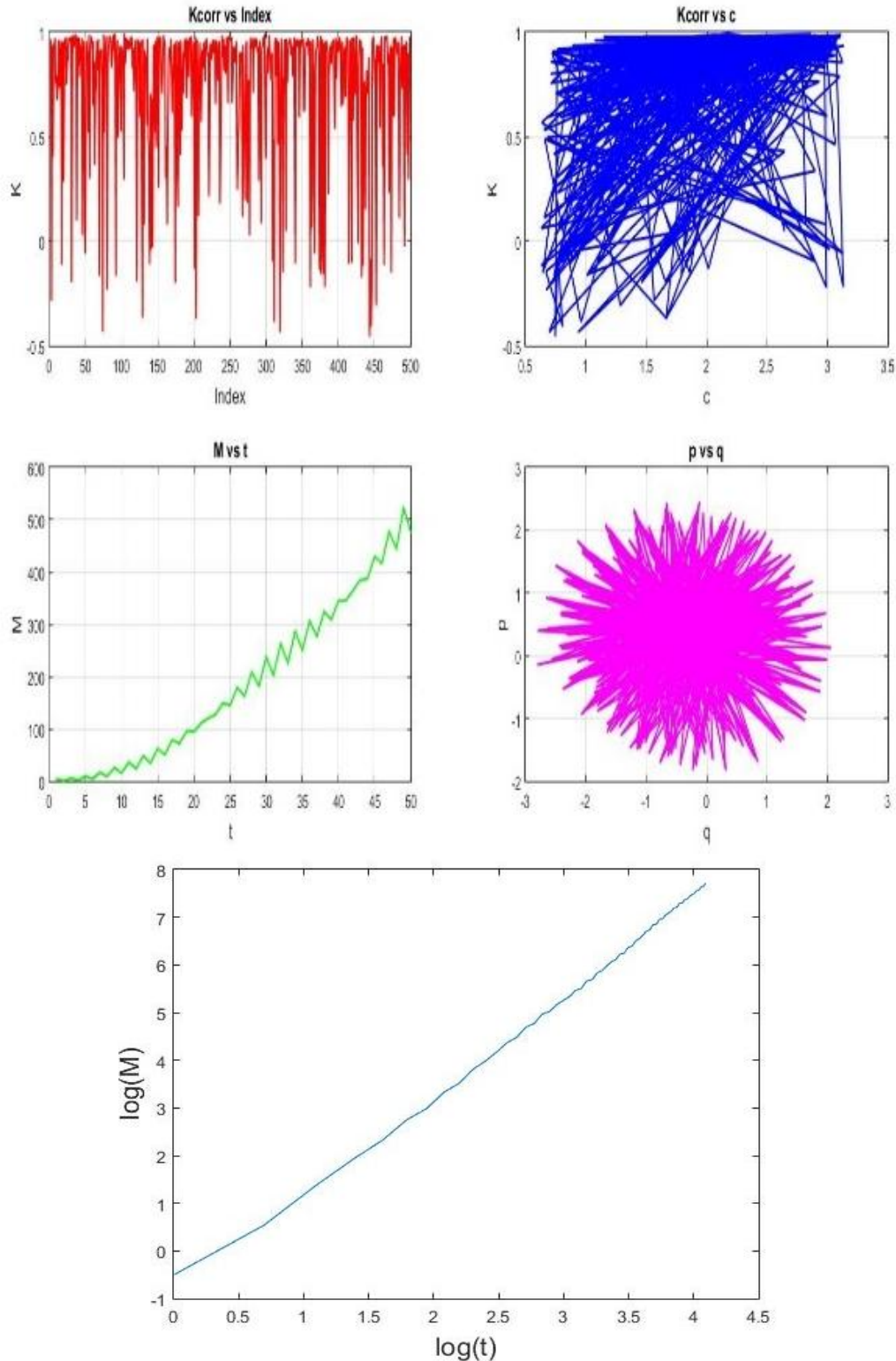
Then,  $D(n) = M(n) - Vosc(n)$

$$K_{corr} = Kc = \lim_{n \rightarrow \infty} \frac{\log Mc(n)}{\log(n)}$$

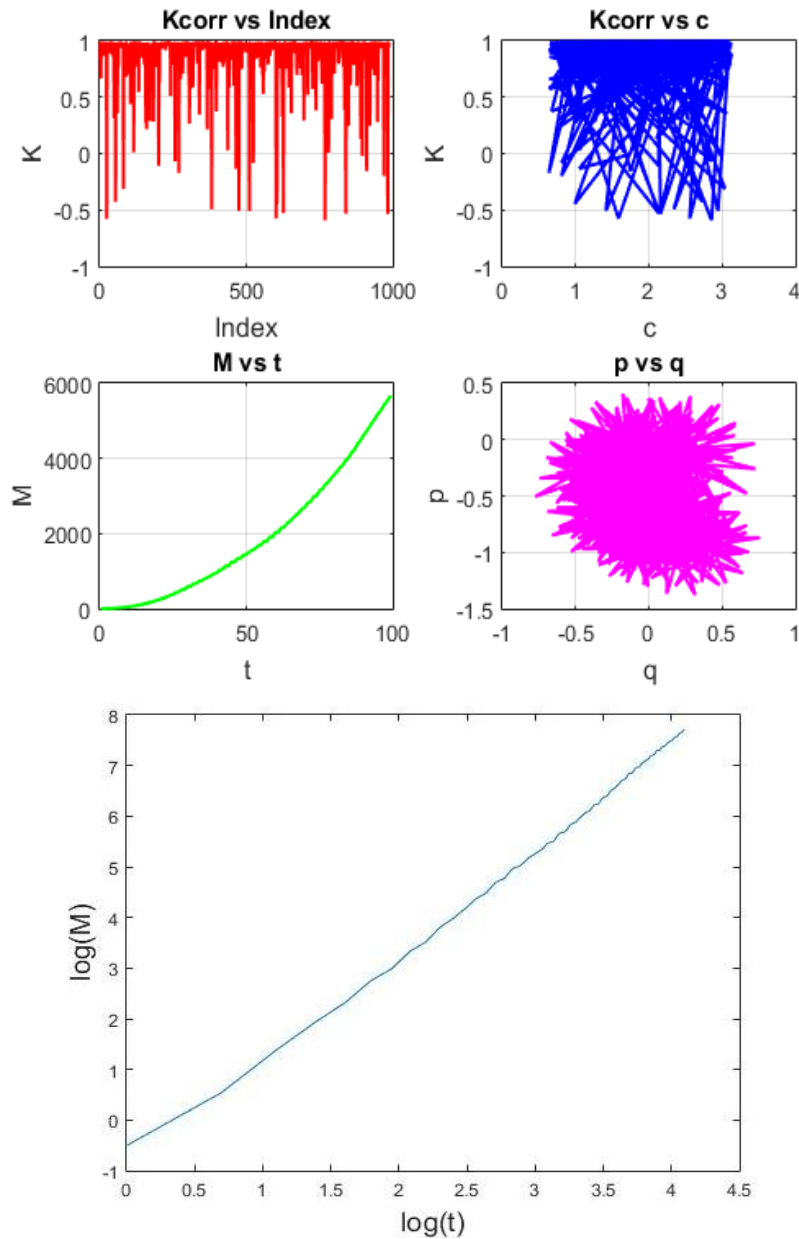
Step (3): Estimate Growth Rate  $K$ : Determine the asymptotic growth rate  $K$  of  $M(n)$ ,  $Kc$  states: If  $Kc \cong 0$  the dynamic is regular (periodic or quasi-periodic). And if  $Kc \not\cong 1$  then the dynamic is chaotic. where  $Kc \in [0, 1]$ .

By using MATLAB program, we get for simulation data:  $Kc = Kcorr = 0.9385 \cong 1$ . Then the system is a chaotic, getting for real data:  $Kc = Kcorr = 0.96148 \cong 1$  it is chaotic too.

The 0–1 test result obtained from the fractional Rössler simulations exhibit a strong agreement with those taken from the real EEG data, show a high level of similarity in their chaotic signatures. This close correspondence confirms the capability of the fractional model to reliably reproduce the underlying dynamical characteristics of the measured brain signals. the figure (6) a&b shows the compared between real data of (EEG)signal and simulation data of FOS (1).



(a) Real Data



(b) Simulation Data

**Figure 6:** Show the Chaotic of (a) Real Data and (b) Simulated Data for the Fractional-Order 4D new System (1) by Zero-One test.

## 7. Conclusion

This study addressed the development of a three-dimension Rössler system into a four-dimension fractional-order model using the Caputo derivative, aiming for a more realistic representation of complex dynamic behaviors. A sinusoidal nonlinearity was introduced with the new term  $\sin(w)$  to enhance the complexity of dynamic behavior. The presence of infinitely many Equilibrium points constitutes strong indicator for existence of hidden attractors, and the system's stability was analyzed using a Jacobi matrix, in addition to the stability condition specific to fractional system, namely the Matignon condition (Yu et al., 2009). The system was found to be unstable, and its dissipative property was confirmed. Numerical simulations using the FDE12 numerical method showed that the system generates diverse behavioral patterns, including chaos and hyper chaos. This was confirmed by Lyapunov's asymptotes and bifurcation diagrams. Fractional order was shown to be a controlling parameter for the system's behavior. To verify this result, the 0-1 chaos test was applied to the simulation data as well as to real EEG data from epilepsy patients. The results showed a clear and strong agreement

between the two behaviors, enhancing the model's ability to represent complex neural systems.

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