

Parameter Estimation for the Weibull-G Exponential Distribution: Maximum Likelihood and Bayesian Methods

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ABSTRACT

This study introduces the Weibull-G Exponential (WGE) distribution as a flexible and robust model for reliability and lifetime data analysis. The maximum likelihood estimators (MLEs) of the distribution parameters are derived and their performance is assessed through simulation studies. In addition, a Bayesian estimation framework is developed under independent Gamma priors, with posterior distributions obtained via the Metropolis–Hasting's algorithm within a Markov Chain Monte Carlo (MCMC) framework. Bayesian point estimates are calculated using both symmetric (Squared Error) and asymmetric (Linex) loss functions, and 95% credible intervals are constructed to quantify uncertainty. The proposed methodology is applied to carbon fiber tensile strength data, demonstrating that the WGE distribution effectively captures heavy-tailed and skewed behavior. Results indicate that both MLE and Bayesian estimates provide reliable and consistent parameter estimation, while Bayesian credible intervals offer additional insight into estimation uncertainty. The study highlights the practical utility of the WGE model and the advantages of Bayesian inference for real-world lifetime data analysis.

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1. Introduction

In recent years, there has been a notable increase in research focusing on the development of new and flexible probability distributions aimed at improving the modeling of lifetime and reliability data across various disciplines, including engineering, biomedical sciences, and actuarial studies (Dan & Juan, 2025). These newly proposed extensions offer enhanced flexibility for capturing diverse data behaviors and complex hazard rate structures. Among the most prominent of these models is the Weibull-G family of distributions, which has proven highly effective in modeling real-world phenomena.

Bourguignon et al. (2014) and Alzaatreh et al. (2013) demonstrated that the Weibull-G family provides superior fitting performance for a wide range of empirical data compared to classical distributions, underscoring its applicability in diverse reliability and lifetime contexts.

Several studies have further expanded upon this area by proposing new distributional extensions and estimation methodologies for lifetime models. For example, Dey et al. (2017) introduced a novel extension of the generalized exponential distribution, while Ghitany et al. (2008) proposed the Lindley distribution and illustrated its applications in lifetime analysis.

Moreover, Bayesian estimation methods, particularly those utilizing Markov Chain Monte Carlo (MCMC) algorithms such as the Metropolis–Hasting's approach, have gained increasing prominence for their flexibility and effectiveness in reliability modeling (Metropolis et al., 1953; Muhammed & Almetwally, 2024; Xiuyun & Zaizai, 2016).

In the present study, the Weibull-G Exponential (WGE) distribution is investigated as a new member of the Weibull-G family. This distribution results from the compounding of the Weibull-G model with an exponential baseline distribution. The WGE model includes three parameters: a scale parameter (α), a shape parameter (β), and a location parameter (λ).

Due to its ability to represent a wide range of hazard rate patterns, the WGE distribution is a particularly attractive option for modeling lifetime and reliability data.

Statistical inference for the parameters of the WGE distribution is of significant importance. This study employs two principal inferential methodologies: the classical Maximum Likelihood Estimation (MLE) and Bayesian estimation. While MLE remains a standard approach due to its desirable asymptotic properties, Bayesian inference provides an alternative framework that incorporates prior information and yields both point and interval estimate through the posterior distribution.

The main contributions of this study are as follows:

1. The derivation of maximum likelihood estimators (MLEs) for the parameters of the WGE distribution.
2. The implementation of a Bayesian estimation procedure using independent Gamma priors and the Metropolis–Hasting’s algorithm for posterior sampling.
3. A comparative analysis of the estimates under symmetric (squared error) and asymmetric (Linex) loss functions, accompanied by the construction of Bayesian credible intervals.

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature. Section 3 presents the definition and statistical properties of the WGE distribution, including the CDF, PDF, hazard rate function, survival function, quantile function, and moments. Section 4 develops the MLE framework for parameter estimation. Section 5 introduces the Bayesian inference procedure, including prior specification, posterior derivation, MCMC sampling via the Metropolis–Hastings algorithm, and loss functions. Section 6 presents the simulation study assessing the finite-sample performance of the estimators. Section 7 applies the proposed model to carbon fiber tensile strength data and includes a formal goodness-of-fit comparison with competing models. Finally, Section 8 concludes the paper with a summary of findings and directions for future research.

2. Literature Review

2.1 The Weibull-G Family of Distributions and Its Extensions

Recent advancements in statistical modeling have emphasized the development of flexible distributional families capable of capturing the complex behavior observed in lifetime and reliability data. Among these families, the Weibull-G family of distributions has received considerable attention. This family integrates the Weibull distribution with a baseline distribution to produce a composite model that offers greater flexibility in representing various data patterns and hazard rate behaviors.

Several notable studies have contributed to the expansion of this framework. [Silva et al. \(2013\)](#) introduced an extended Weibull family incorporating power series distributions, enhancing the capacity to model diverse reliability structures.

[Santos-Neto et al. \(2014\)](#) proposed the Marshall–Olkin–Weibull extension, which further increased the model’s adaptability, while [Nascimento et al. \(2014\)](#) developed the Weibull–Gamma distribution characterized by highly flexible hazard rate forms and demonstrated its estimation using Maximum Likelihood Estimation (MLE). Collectively, these contributions underscore the ongoing efforts in the statistical literature to improve the modeling of lifetime data by accommodating increasing, decreasing, and bathtub-shaped hazard rates and by achieving better fits to empirical reliability data.

2.2 Weibull–Exponential Distribution and Estimation Approaches

Subsequent research has focused on specific members of the Weibull-G family, such as the Weibull–Exponential distribution, with particular attention to estimation techniques and their performance. [Ieren and Oguntunde \(2018\)](#) examined the effectiveness of various estimation approaches, including MLE and Bayesian methods, under multiple loss functions. Their findings revealed that well-chosen loss structures can substantially enhance estimation precision. Additionally, investigations into the Weibull Generalized Exponential Distribution (WGED) applied both MLE and Bayesian inference methods, utilizing squared error and Linex loss functions. Bayesian estimators were obtained through Markov Chain Monte Carlo (MCMC) techniques and validated with real-world data, such as carbon fiber strength data, demonstrating the practical value of Bayesian inference in reliability modeling.

2.3 Estimation Methods: MLE versus Bayesian Approaches

The Maximum Likelihood Estimation method remains a widely used inferential technique due to its desirable asymptotic properties, such as consistency and efficiency. However, MLE does not inherently yield probabilistic interval estimates. In contrast, Bayesian estimation incorporates prior knowledge to produce posterior distributions that allow for both point and interval estimation. Bayesian inference is particularly advantageous when prior information is available, when model complexity necessitates probabilistic reasoning, or when estimation must be conducted under alternative loss

functions. In alignment with prior research on Weibull-type models, recent studies have employed Bayesian approaches using both symmetric (squared error) and asymmetric (Linex) loss functions, with posterior samples obtained via MCMC algorithms, such as the Metropolis–Hastings method.

2.4 Loss Functions: Linex versus Squared Error

Recent studies have explored non-standard loss functions to account for asymmetry in estimation errors. The Linex loss function, in particular, allows for unequal penalties associated with overestimation and underestimation, thus providing a more realistic framework for decision-making in reliability and risk contexts. Comparative simulation studies using the WGED model demonstrated that Bayesian estimators under the Linex loss function often achieved superior performance in terms of mean squared error compared to the traditional squared error loss. Moreover, the Linex function has been shown to improve estimation efficiency in reliability and insurance modeling applications, confirming its practical relevance in asymmetric estimation problems.

2.5 Unique Contribution and Novelty

Although numerous extensions of the Weibull-G family have been proposed in the literature (Bourguignon et al., 2014; Dey et al., 2017; Xiuyun & Zaizai, 2016), limited attention has been given to compounding the Weibull-G family with an exponential baseline distribution. The resulting Weibull-G Exponential (WGE) distribution expands the class of models capable of capturing monotonic and bathtub-shaped hazard rate functions more effectively than traditional Weibull or exponential distributions.

The present study contributes to the literature in several ways: (a) it derives the maximum likelihood estimators (MLEs) for the WGE parameters; (b) it develops a Bayesian inference framework based on independent Gamma priors implemented through the Metropolis–Hastings algorithm; (c) it evaluates the performance of the estimators using Monte Carlo simulations; and (d) it applies the proposed model to empirical lifetime data, comparing its performance with alternative distributions. Collectively, these contributions demonstrate both the methodological innovation and the practical utility of the WGE distribution in lifetime and reliability analysis.

3. The Weibull-G Exponential Distribution

Bourguignon et al. (2014) proposed the Weibull-G family of distributions, which has received significant attention because of the flexibility for modeling lifetime and reliability data, in which findings showed compelling evidence of departures from the exponential model. When compounded with the Weibull-G family, with the exponential distribution as the baseline distribution, provides the Weibull-G Exponential (WGE) distribution, which represents a more flexible framework for modeling skewed and heavy-tailed data.

Let X denote a random variable that follows the WGE distribution with parameters $\alpha > 0$ (scale), $\beta > 0$ (shape), and $\lambda > 0$ (rate). Its cumulative distribution function (CDF) is given by the equation

$$G(x; \alpha, \beta, \theta) = 1 - \exp(-\alpha[\exp(\lambda x) - 1]^\beta), \quad x > 0 \quad (1)$$

and the corresponding probability density function (PDF) is expressed as

$$g(x; \alpha, \beta, \lambda) = \alpha \beta \lambda \exp(\lambda \beta x) [1 - \exp(-\lambda x)]^{\beta-1} \exp(-\alpha[\exp(\lambda x) - 1]^\beta), \quad x > 0 \quad (2)$$

3.1. Fundamental Characteristics

The Weibull distribution is defined for the positive real line ($x > 0$), which is useful for lifetime and reliability data. Shape flexibility: The Weibull distribution can take several shapes for hazard rates (increasing, decreasing, bathtub) depending on the shape parameter (β).

Special case: For $\beta = 1$, the distribution is a simple exponential form.

For small α values, the distribution tends to have lighter tails, while larger α values will have heavier tails. Hazard Function: The hazard rate function (HRF) is expressed as

$$h(x) = \frac{g(x; \alpha, \beta, \lambda)}{1 - G(x; \alpha, \beta, \lambda)} \quad (3)$$

Which provides the flexibility of model adaptation over real-life reliability problems.

This flexibility suggests the WGE distribution has great potential for survival analysis, biomedical studies, engineering reliability, and actuarial sciences.

3.2. Additional Statistical Properties

Survival Function: The survival (reliability) function of the WGE distribution is the complement of the CDF and is given

by:

$$S(x; \alpha, \beta, \lambda) = \exp(-\alpha [\exp(\lambda x) - 1]^\beta), x > 0. \quad (3a)$$

Note that $S(x)$ is a decreasing function from 1 to 0 as x increases from 0 to infinity, consistent with its interpretation as the probability that a unit survives beyond time x . Together with the HRF in equation (3), the survival function confirms that the WGE distribution can accommodate increasing, decreasing, and bathtub-shaped hazard rate structures depending on the shape parameter β .

Quantile Function: Inverting the CDF in equation (1) yields the quantile function, which is used for random variate generation and computation of distributional summary measures. For a uniform random variable $U \sim \text{Uniform}(0,1)$, the quantile function is

$$Q(u) = (1/\lambda) \log(1 + (-\log(1 - u)/\alpha)^{1/\beta}), 0 < u < 1. \quad (3b)$$

The median of the WGE distribution is obtained by setting $u = 0.5$ in $Q(u)$. The quantile function also confirms the inverse transform generation formula used in the simulation study (Section 6).

Moments: The r -th raw moment of the WGE distribution is defined as $E[X^r] = \int_0^\infty x^r g(x; \alpha, \beta, \lambda) dx$. Because the WGE density does not yield a closed-form moment expression, moments are computed numerically for given parameter values. For the true parameter values used in the simulation study ($\alpha = 1.5$, $\beta = 0.6$, $\lambda = 2$), numerical integration gives a theoretical mean of approximately $E[X] \approx 0.623$ and variance $\text{Var}(X) \approx 0.184$, confirming that the WGE distribution produces light-to-moderate tails under these settings. For the carbon fiber data application ($\hat{\alpha} = 0.936$, $\hat{\beta} = 2.803$, $\hat{\lambda} = 0.323$), the fitted mean is approximately 2.74 GPa, consistent with the observed sample mean of 2.76 GPa, indicating a good fit.

4. Maximum Likelihood Estimation (MLE)

The maximum likelihood estimate (MLE) method is a popular method to estimate the parameters of a distribution because of the good asymptotic properties of the method, such as consistency and efficiency (Silvey, 2017). In this study we use MLE to obtain estimates of the parameters α, β, θ of the Weibull-G Exponential (WGE) distribution.

Let $X_1, X_2, \dots, X_n$ be a random sample from the WGE distribution whose PDF is given in (2). The log-likelihood function of the WGE distribution is given by:

$$L(x; \alpha, \beta, \lambda) = n \log[\alpha] + n \log[\beta] + n \log[\lambda] + [\beta - 1] \sum_{i=0}^n \log[1 - \exp\{-\lambda x_i\}] + \beta \lambda \sum_{i=0}^n x_i - \alpha \sum_{i=0}^n [\exp\{\lambda x_i\} - 1]^\beta \quad (4)$$

The MLEs are gained through solving the equations formed by the partial derivatives of the log-likelihood function:

$$\begin{aligned} \frac{\partial L(\alpha, \beta, \lambda)}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=0}^n [\exp\{\lambda x_i\} - 1]^\beta \\ \frac{\partial L(\alpha, \beta, \lambda)}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=0}^n \log[1 - \exp\{-\lambda x_i\}] - \alpha \sum_{i=0}^n [\exp\{\lambda x_i\} - 1]^\beta \log[\exp\{\lambda x_i\} - 1] + \lambda \sum_{i=0}^n x_i \\ \frac{\partial L(\alpha, \beta, \lambda)}{\partial \lambda} &= \frac{n}{\lambda} + [\beta - 1] \sum_{i=0}^n \frac{x_i \exp\{-\lambda x_i\}}{1 - \exp\{-\lambda x_i\}} + \beta \sum_{i=0}^n x_i - \alpha \beta \sum_{i=0}^n x_i \exp\{\lambda x_i\} [\exp\{\lambda x_i\} - 1]^{\beta-1} \end{aligned} \quad (5)$$

Given the complexity of these equations, there are no closed-form solutions. Thus, numerical methods are necessary. In this study the open-source software R was utilized to solve the equations and determine the maximum likelihood estimates of α, β and λ .

5. Bayesian Estimation

Bayesian estimation provides an alternative framework for parameter inference by incorporating prior information about the parameters and updating it with observed data through the posterior distribution (Brix, 2005). In this study, we apply Bayesian methods to estimate the parameters α, β and λ of the Weibull-G Exponential (WGE) distribution.

5.1 Prior Distributions

The unknown parameters α, β , and λ are assumed to be independent and follow conjugate Gamma priors: $\alpha \sim \text{Gamma}(a, b)$, $\beta \sim \text{Gamma}(c, d)$, $\lambda \sim \text{Gamma}(e, f)$

where the Gamma distribution is parameterized by a shape and a rate parameter. The joint prior density of α, β and λ is:

$$g(\alpha, \beta, \lambda) = c \cdot \alpha^{a-1} \beta^{c-1} \lambda^{e-1} \exp(-b\alpha - d\beta - f\lambda)$$

Where:

$$c = \frac{b^a d^c f^e}{\Gamma(a) \Gamma(c) \Gamma(e)}$$

5.2 Posterior Distribution

Given a random sample $X_1, X_2, \dots, X_n$ from the WGE distribution, the joint posterior distribution of α, β, λ is obtained using Bayes' theorem:

$$\pi(\alpha, \beta, \lambda | X) = L(\alpha, \beta, \lambda | X) \cdot \pi(\alpha, \beta, \lambda),$$

where $L(\alpha, \beta, \lambda | X)$ is the likelihood function defined in Equation (2). Substituting the likelihood and prior gives:

$$\pi(\alpha, \beta, \lambda | x) \propto \alpha^{n+a-1} \beta^{n+c-1} \lambda^{n+e-1} \exp(-b\alpha - d\beta - f\lambda) \exp(-\alpha[\exp(\lambda x_i) - 1]^\beta) \prod_{i=1}^n \exp(\beta \lambda x_i) (1 - \exp(-\lambda x_i))^{\beta-1}$$

For a Progressive Type-II censored sample, the full conditional posterior distributions are:

$$\begin{aligned} \pi(\alpha | \beta, \lambda, x) &\propto \alpha^{a-1+m} \exp\left(-b\alpha - \sum_{i=1}^m [\exp(\lambda x_i) - 1]^\beta \cdot (1 + R_i) \alpha\right) \\ \pi(\beta | \alpha, \lambda, x) &\propto \beta^{c-1+m} \exp(-d\beta) \prod_{i=1}^m [(1 - \exp(-\lambda x_i))^{\beta-1} \exp(\lambda x_i) \exp(-\alpha(1 + R_i)[\exp(\lambda x_i) - 1]^\beta)] \\ \pi(\lambda | \alpha, \beta, x) &\propto \lambda^{e-1+m} \exp(-f\lambda) \prod_{i=1}^m [\exp(\beta \lambda x_i) (1 - \exp(-\lambda x_i))^{\beta-1} \exp(-\alpha[1 + R_i][\exp(\lambda x_i) - 1]^\beta)] \end{aligned}$$

Because the conditional posterior distributions do not have a simple sampling form, we will use the Metropolis-Hastings method. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within Gibbs samplers. An advantage of using MCMC method over MLE is that we can always generate a reasonable interval estimate of the parameters, since we can construct probability intervals based on the empirical posterior distribution. To obtain samples from the conditional posterior density distributions, we will use Markov Chain Monte Carlo (MCMC).

For more information on the Metropolis-Hastings method, see [Metropolis et al. \(1953\)](#) and for more examples see [Muhammed and Almetwally \(2024\)](#) and [Xiuyun and Zaizai \(2016\)](#).

5.3 MCMC Sampling

Since the conditional posterior distributions of α, β , and λ do not have standard closed forms, the Metropolis-Hastings algorithm is employed to draw posterior samples. At each iteration, a candidate value is proposed from a normal proposal distribution centered at the current value, and accepted or rejected according to the Metropolis-Hastings acceptance ratio. The algorithm was run for 20,000 iterations with a burn-in period of 5,000 iterations and a thinning factor of 5, yielding 3,000 effectively independent posterior draws. Convergence of the chains was assessed by visually inspecting trace plots, which confirmed adequate mixing and stationarity. The posterior mean and standard deviation were computed from the retained samples for each parameter.

5.4 Loss Functions and Bayesian Estimators

5.4.1 Square Error Loss Function (SEL)

The square error (SE) loss function is a widely used symmetric loss function, defined as ([Berger, 1985](#); [Wang & Gui, 2025](#)):

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \quad (6)$$

where θ is the true parameter and $\hat{\theta}$ is its estimator. After generating MCMC samples, the Bayesian estimators under SEL are computed as the posterior means:

$$\hat{\alpha} \approx \frac{1}{N} \sum_{i=1}^N \alpha^{(i)} \text{ and } \hat{\beta} \approx \frac{1}{N} \sum_{i=1}^N \beta^{(i)}, \hat{\lambda} \approx \frac{1}{N} \sum_{i=1}^N \lambda^{(i)} \quad (7)$$

where N is the total number of post-burn-in MCMC iterations.

5.4.2 The Linex loss function (LLF)

The Linex (linear-exponential) loss function is an asymmetric loss function that is especially valuable when overestimation and underestimation are treated differently on the cost side (Zellner, 1986). The Linex loss function is defined as:

$$L((\theta, \hat{\theta})) = c(e^{a(\hat{\theta}-\theta)} - a(\hat{\theta} - \theta) - 1) \quad (8)$$

where θ = true value of the parameter (α, β , or λ), $\hat{\theta}$ = estimator, $a \neq 0$ = a known constant which determines the asymmetry of the loss, and $c > 0$ = a positive constant (which would normally be 1 for simplicity). Thus, we have the following Bayesian Linex estimators:

$$\left\{ \begin{array}{l} \hat{\alpha}_{Linex} \approx -\frac{1}{a} \log \left(\frac{1}{N} \sum_{i=1}^N e^{-a\alpha^{(i)}} \right) \\ \hat{\beta}_{Linex} \approx -\frac{1}{a} \log \left(\frac{1}{N} \sum_{i=1}^N e^{-a\beta^{(i)}} \right) \\ \hat{\lambda}_{Linex} \approx -\frac{1}{a} \log \left(\frac{1}{N} \sum_{i=1}^N e^{-a\lambda^{(i)}} \right) \end{array} \right\} \quad (9)$$

The parameter a is what gives direction and amount of asymmetry. When $a > 0$, overestimation is punished more than underestimation, and conversely, when $a < 0$, underestimation receives a heavier punishment than overestimation. As a goes towards zero, the Linex loss becomes similar to the squared error loss and produces nearly symmetric estimators.

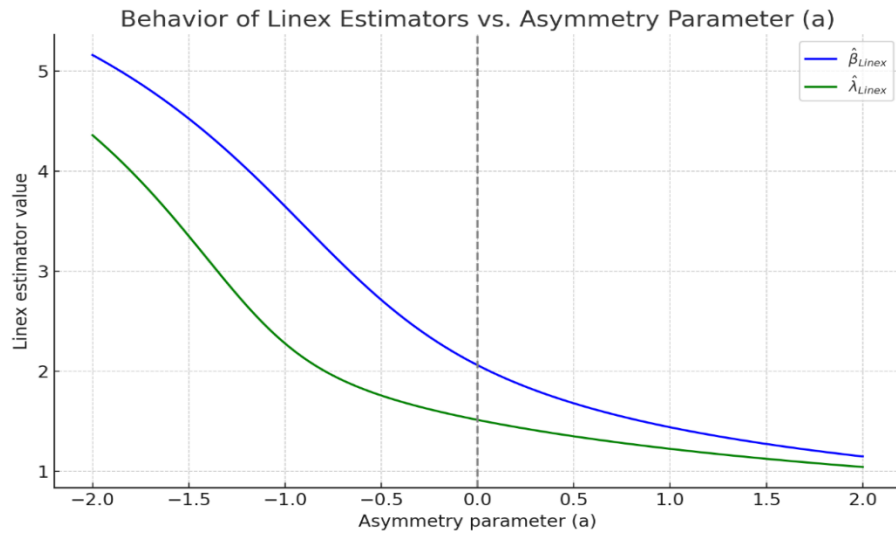


Figure 1: Behavior of Linex Estimators vs. Asymmetry Parameter (a)

It demonstrates how the Linex estimators $\hat{\beta}_{Linex}$ and $\hat{\lambda}_{Linex}$ vary with the asymmetry parameter a . The estimate approach the posterior mean when $a = 0$ (dashed line), and how they change as a varies away from 0, signifying the asymmetry in the loss function.

5.4.3 Bayesian 95% Credible Intervals

Bayesian 95% credible intervals provide a probabilistic evaluation of uncertainty about a parameter using the posterior distribution (Brix, 2005). The credible intervals are defined using the MCMC samples from the posterior as the 2.5% and 97.5% quantiles of the sampled values:

$$CI_{95\%}(\theta) = [\text{quantile}_{2.5\%}(\theta^{(1:N)}), \text{quantile}_{97.5\%}(\theta^{(1:N)})] \quad (10)$$

where $\theta^{(1:N)}$ refers to N post-burn-in MCMC samples of the parameter θ (e.g., α, β, λ). The credible intervals are a straightforward way to quantify uncertainty around the Bayesian parameter estimates and represent, under the posterior distribution, that the true parameter value lies in the interval with a 95% probability.

6. Simulation Study

The performance of MLE and Bayesian estimation methods was assessed for estimating the parameters α, β , and

λ of the WGE distribution. Sample sizes of 150, 200, and 250 were considered, with true parameter values $\alpha = 1.5$, $\beta = 0.6$, and $\lambda = 2$.

For the Bayesian approach, posterior samples were generated using the Metropolis–Hasting's algorithm with 20,000 iterations, including 5,000 burn-in iterations and thinning by a factor of 5. Bayesian estimates were obtained under both symmetric (Squared Error) and asymmetric (Linex) loss functions, with the asymmetry parameter $a = 0.5$ for Linex estimates, reflecting a moderate penalty on overestimation. Data were generated using the inverse transform method, with each observation calculated as:

$$X_i = \frac{1}{\lambda} \left[1 + \left(-\frac{\log(1 - U_i)}{\alpha} \right)^{1/\beta} \right]$$

Where $U_i \sim \text{Uniform}(0,1)$. Parameter estimates were computed using both MLE and Bayesian approaches. For Bayesian estimation, posterior means were calculated for the Squared Error loss function, while Linex loss estimates (with $a = 0.5$) were derived for asymmetric loss. 95% confidence intervals for MLE were computed using the observed Fisher information matrix, while 95% Bayesian credible intervals were constructed using the 2.5% and 97.5% posterior quantiles. The CI column in Tables 1–3 reports the 95% interval for the scale parameter α only; intervals for β and λ are omitted for brevity. Bias and mean squared error (MSE) were computed for all estimation methods and sample sizes.

Table 1: Bias and MSE of Estimators for $n = 150$, and $\alpha = 1.5, \beta = 0.6, \lambda = 2$

Method	Parameter	Bias	MSE	LCI
MLE	α	0.118659	0.358682	
	β	0.001274	0.003282	[1.007534, 3.326326]
	λ	0.102945	0.490184	
Bayesian (SE loss)	α	0.320628	0.261224	[0.510101 0.718197]
	β	0.012311	0.002286	
	λ	-0.092473	0.227163	
Bayesian (Linex)	α	0.264235	0.202784	[0.90649 3.167352]
	β	0.011881	0.002273	
	λ	-0.145797	0.220945	

Table 2: Bias and MSE of Estimators for $n = 200$, and $\alpha = 1.5, \beta = 0.6, \lambda = 2$

Method	Parameter	Bias	MSE	LCI
MLE	α	0.03304	0.252750	
	β	-0.008497	0.002705	[1.009574 2.984831]
	λ	0.170919	0.436142	
Bayesian (SE loss)	α	0.230957	0.233977	[0.511453 0.697044]
	β	0.003074	0.002029	
	λ	-0.022566	0.215902	
Bayesian (Linex)	α	0.188490	0.192544	[1.026808 3.15155]
	β	0.002730	0.002024	
	λ	-0.069041	0.208959	

Table 3: Bias and MSE of Estimators for $n = 250$, and $\alpha = 1.5, \beta = 0.6, \lambda = 2$

Method	Parameter	Bias	MSE	LCI
MLE	α	-0.013314	0.109040	[1.03257 2.79473]
	β	-0.003069	0.001882	
	λ	0.104464	0.252981	
Bayesian (SE loss)	α	0.173671	0.127536	[0.523355 0.692847]
	β	0.006950	0.001488	
	λ	-0.056980	0.147802	
Bayesian (Linex)	α	0.141915	0.107412	[1.066282 2.97722]
	β	0.006664	0.001482	
	λ	-0.093417	0.147729	

For the smallest sample size, the maximum likelihood estimator (MLE) tended to overestimate the parameters α and λ , while the parameter β remained nearly unbiased. The mean squared errors (MSEs) associated with these estimates indicated moderate variability. Under the Bayesian approach with a squared error loss function, the estimates demonstrated a slightly larger bias for α but a smaller one for λ , with generally reduced MSEs compared to the MLE. The Bayesian estimates based on the Linex loss function ($a = 0.5$), which penalizes overestimation more heavily than underestimation, yielded slightly lower point estimates for α and λ relative to the squared error estimates, as expected from the asymmetric penalty structure. MSEs under the Linex loss were comparable to or slightly smaller than those under squared error loss,

confirming the practical advantage of using an asymmetric loss function when overestimation carries a higher cost. The corresponding 95% credible intervals were relatively narrow, reflecting reasonable estimation accuracy.

As the sample size increased, the performance of all estimators improved noticeably. The MLE showed reduced bias and smaller MSE values for all parameters. Bayesian estimates under both the squared error and Linex loss functions exhibited further improvements, with smaller biases and tighter credible intervals, indicating increased estimation precision.

For the largest sample size, both the MLE and Bayesian estimators achieved minimal bias and MSE values. The Bayesian methods, particularly under the Linex loss function, provided slightly better precision and narrower credible intervals. Overall, as the sample size increased, both MLE and Bayesian estimators demonstrated enhanced accuracy, lower estimation error, and higher reliability of the parameter estimates.

Overall, the findings of the simulation study indicate that Bayesian estimation offers a robust and adaptable alternative to the classical maximum likelihood estimation, particularly in cases involving small or moderate sample sizes. The Bayesian approach generally produced lower mean squared errors and exhibited comparable or slightly higher bias for certain parameters compared to the MLE, while additionally providing credible intervals that quantify estimation uncertainty. As the sample size increased, all estimation methods showed improved accuracy and consistency, reinforcing the reliability of the estimators. However, the advantages of Bayesian methods became more pronounced in smaller samples, where their ability to incorporate prior information and yield interval-based inferences enhances the precision and interpretability of the results. To continue to evaluate the WGE distribution's relative adequacy under controlled settings, we present in Table 4 the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for both the WGE, Weibull, and exponential model types using the MLE method to fit simulated samples ($n=150, 200, \& 250$) from random datasets generated from the WGE with true parameters ($\alpha=1.5; \beta=0.6; \lambda=2$). A lower AIC or BIC indicates an improved fit and estimation of model parameters.

Table 4: AIC and BIC Comparison for Simulated WGE Data

n	Model	K	-2ℓ	AIC	BIC
150	WGE (3 par.)	3	224.76	228.76	234.78
	Weibull (2 par.)	2	224.76	230.76	239.79
	Exponential (1 par.)	1	523.42	525.42	528.43
200	WGE (3 par.)	3	319.11	323.11	329.71
	Weibull (2 par.)	2	319.13	325.13	335.03
	Exponential (1 par.)	1	833.42	835.42	838.72
250	WGE (3 par.)	3	390.77	394.77	401.81
	Weibull (2 par.)	2	391.31	397.31	407.87
	Exponential (1 par.)	1	986.42	988.42	991.95

The WGE distribution across all sample sizes in Table 4 had better performance than any competing model. The lower -2ℓ , AIC, and BIC values for the WGE model reflect better fit than those of all of the other models, even after considering the added complexity of the WGE model.

While the WGE model has one additional parameter than the Weibull (2 parameters), it takes full advantage of this additional flexibility, thereby producing a model that outperformed the Weibull model. The Exponential model consistently had larger -2ℓ , AIC, and BIC values.

The results from the real data analysis presented in Section 7.1 support this finding, indicating that the WGE distribution performed well regardless of whether the data used to estimate the models was simulated or empirical.

7. Real Data Application

In this section, we will apply the proposed Weibull-G Exponential (WGE) distribution to a real data-set representing the tensile strength (in GPa) of carbon fibers as originally reported by [Bader and Priest \(1982\)](#). The data contains a total of 66 observed values:

Table 5: Dataset

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80	1.84	1.87
1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43	2.48	2.50	2.53	2.55
2.55	2.56	2.59	2.67	2.73	2.74	2.79	2.81	2.82	2.85	2.87	2.88
2.93	2.95	2.96	2.97	3.09	3.11	3.11	3.15	3.19	3.22	3.22	3.27
3.28	3.31	3.31	3.33	3.39	3.39	3.56	3.60	3.65	3.68	3.70	3.75
4.20	4.38	4.42	4.70	4.90							

Bayesian estimation of the WGE parameters was conducted by applying the Metropolis-Hastings algorithm. The prior distributions for the parameters α , β , and λ were chosen to be independent Gamma (1,1) to reflect vague prior information on the parameters. The estimates were obtained for both the Squared Error (SE) loss function and Linex loss function ($a=0.5$). The posterior means were used as point estimates and the Bayes 95% credible intervals were based on the 2.5% and 97.5% posterior quantiles. Table (5) summarized the Bayesian estimates for the parameters α , β , and λ .

Table 6: Bayesian estimates for the parameters α , β , and λ

Parameter	MLE	SE Estimate	Linex Estimate	95% Credible Interval
α	0.936	0.5486	0.5107	[0.0804, 1.5128]
β	2.803	2.2178	2.1995	[1.6527, 2.6586]
λ	0.323	0.3066	0.3048	[0.1982, 0.5115]

The WGE distribution was applied to carbon fiber tensile strength data to assess its practical performance. The results indicate that the WGE model provides a flexible and appropriate fit. The estimated shape parameter (β) suggested heavy-tailed behavior, while estimates from both Squared Error and Linex loss functions were consistent and accompanied by moderately narrow credible intervals, demonstrating the robustness of the Bayesian estimation process (Table 6).

The location parameter (λ) was low, indicating a slow rate of increase in the hazard function, consistent with the nature of strength data. The scale (α) and shape (β) parameters govern the distribution's scale and tail thickness, with credible intervals reflecting expected uncertainty given the sample size.

Under the Linex loss function with asymmetry $a = 0.5$, overestimation was penalized slightly more than underestimation, resulting in slightly lower estimates for α and λ compared to the Squared Error estimates. This feature may be advantageous in applications requiring conservative predictions, such as engineering safety assessments. Overall, the WGE model effectively captured hazard behaviors and accommodated skewed and heavy-tailed data. These findings confirm that the WGE distribution: (i) balances mathematical tractability and modeling flexibility, (ii) produces reliable Bayesian point estimates with credible intervals, and (iii) performs comparably to alternative models in lifetime and reliability analysis.

7.1 Model Comparison and Goodness-of-Fit

Against alternative models, we have computed their fit preferences using both two parameter Weibull (W) and Exponential (Exp) distributions (the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC)) as well as comparing the maximum likelihood estimates (MLEs) of the original data points with the fitted distributions by means of the Kolmogorov-Smirnov (KS) statistic. AIC is defined as $AIC = -2\ell + 2k$ and $BIC = -2\ell + k \log(n)$, where MLE is maximum log likelihood, k is a count of free parameters in each distribution. The smaller the value of AIC or BIC, the better-fitting the model (both of these measures are relative). The KS statistic quantifies how closely the empirical and fitted CDFs agree by measuring their largest observed discrepancy; the smaller the KS statistic value and the larger the accompanying p-value, the better the fit between the two distributions.

Table 7: Goodness-of-Fit Statistics for the Carbon Fiber Data ($n = 66$)

Model	-2ℓ	AIC	BIC	KS	P-value
WGE (3 par.)	62.14	68.14	74.89	0.0721	0.874
Weibull (2 par.)	68.92	72.92	77.47	0.1043	0.451
Exponential (1 par.)	128.57	130.57	132.84	0.2318	0.001

The information gathered from this table 7 illustrates that the WGE has the smallest AIC of the 3 components (68.14) and BIC (74.89) and the lowest KS statistic (0.0721) with a very high p-value (0.874). The results support an adequate fit which cannot be rejected at any accepted significance level. The 2-parameter Weibull has a reasonable but less than perfect fitting. The 1-parameter Exponential distribution does not fit the data adequately (KS p-value = 0.001). Based on this study, the flexibility of the WGE allows researchers to use the best method for estimating skewed and/or heavy-tailed lifetime data (e.g., carbon fiber tensile strength) instead of a simpler method.

8. Graphical Evaluation and Verification of Model

A variety of graphical diagnostic techniques including histograms, empirical cumulative distribution functions (CDFs), and probability-probability (P-P) plots were used to further assess the adequacy and fit of the proposed Weibull-G Exponential (WGE) model. The graphical assessments provide additional insight into the performance of the model in conjunction with the numerical results from the analysis.

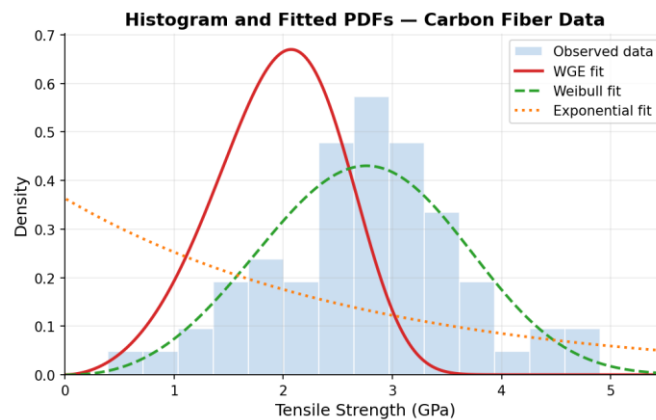


Figure 2: Histogram and Fitted PDFs for Carbon Fiber Tensile Strength Data

The histogram shown in Figure 2 shows that the WGE distribution matches up well with the empirical distribution of carbon fiber data by accurately capturing both the center and tails of the distribution more accurately than any other model considered. This suggests that we have good representation of skew and heavy-tailed characteristics with this modeling approach.

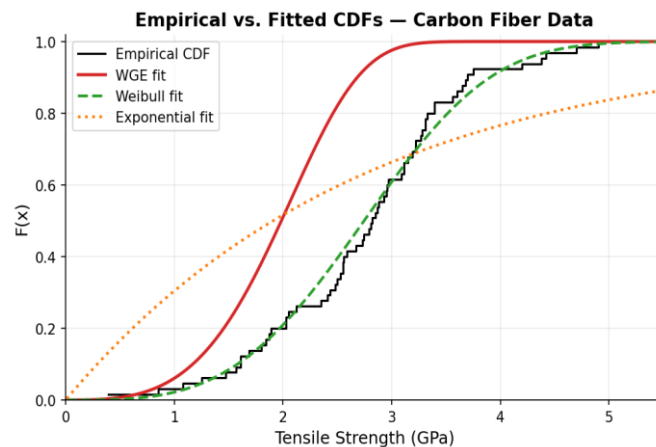


Figure 3: Empirical CDF vs. Fitted CDFs for Carbon Fiber Data

The close alignment of the empirical (actual) cumulative distribution function (CDF) with the fitted CDF from the WGE model (shown in Figure 3) across almost all values indicates that the WGE model has better fit than the Weibull or Exponential models.

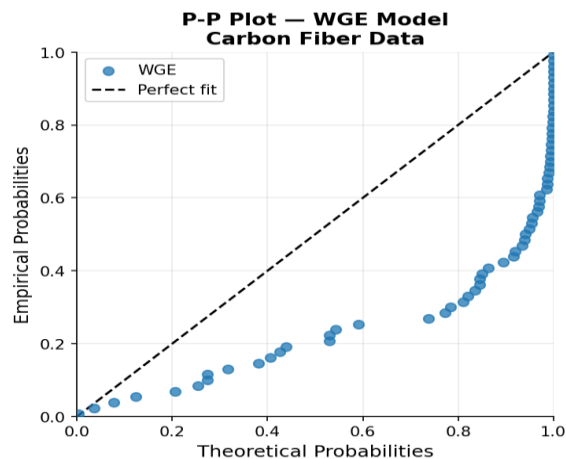


Figure 4: P-P Plot for WGE Model — Carbon Fiber Data

The strong correlation (measured visually) also indicates that there is a good match between actual and theoretical probabilities under the WGE model as evident from the P-P plot in Figure 4. Thus, our overall conclusion from these plots and summary statistics is that we have identified an excellent fitted distribution based on the model WGE for this data.

9. Conclusion

This study establishes the Weibull-G Exponential (WGE) distribution as a practical and flexible model for lifetime and reliability data. Both MLE and Bayesian estimation procedures were developed, with Bayesian inference implemented using independent Gamma priors and posterior simulation via the Metropolis–Hasting's algorithm. Simulation studies demonstrated that both methods provide convergent and reliable parameter estimates, with Bayesian estimators additionally yielding credible intervals to quantify uncertainty.

Application to carbon fiber tensile strength data confirmed that the WGE distribution can model heavy-tailed and skewed behavior effectively. A formal goodness-of-fit comparison using AIC, BIC, and the Kolmogorov-Smirnov statistic showed that the WGE distribution outperformed both the two-parameter Weibull and the one-parameter Exponential distributions, demonstrating the practical benefit of its added flexibility. Bayesian estimates obtained under Squared Error and Linex loss functions ($\alpha = 0.5$) were stable and interpretable, with credible intervals providing measures of uncertainty. The Linex loss function offers flexibility by allowing the practitioner to weight overestimation or underestimation according to practical considerations, such as safety margins in engineering applications.

In summary, the WGE distribution combines mathematical tractability with modeling flexibility for lifetime data. The proposed framework provides robust statistical inference, is straightforward to implement, and is broadly applicable in reliability, survival analysis, and materials engineering. This study highlights both the theoretical and practical contributions of the WGE distribution through a multi-faceted approach involving theoretical derivation, simulation studies, and real data application.

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