

Differential Geometry of the Statistical Manifold and its Submanifold for the Shanker–Mishra Lindley Distribution

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ABSTRACT

This paper presents a comprehensive geometric study of the statistical manifold of the two-parameter Shanker and Mishra Lindley Distribution (S.M.L.D). It first proves that the distribution family forms a smooth two-dimensional statistical manifold parameterized by $E = R^+ \times R^+$ with injective parameter mappings. It then establishes the foundational structure by deriving the Riemannian Fisher metric. g_{ii} , the geodesic equations using Christoffel symbols, and the Hellinger distance between any two distributions. The paper further explores the concept of α -connections and proves the existence of a unique flattening coefficient α^* that transforms the manifold from being curved under the exponential connection ($\alpha = 1$) into a flat manifold. Finally, it examines the special case of fixing the parameter. ω , which reduces the model to a one-parameter exponential family. For this submanifold, the dual coordinates (ϑ, ρ) , the corresponding Fisher metric, the geodesic equations in dual coordinates, and the associated canonical (Bregman) divergence are derived. It aims to provide researchers with advanced tools (such as geometric statistical measures or information geometry) to enhance the accuracy and efficiency of parametric statistical inference.

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1. Introduction

The statistical manifold provides a geometric framework for studying families of parametric probability distributions, where each member (distribution) is represented by a unique point on the manifold. The geometric properties of this manifold (such as the Fisher-Rau scale) are used to describe the internal structure and inherent relationships between different distributions. Information engineering establishes the theoretical foundation for analyzing statistical structures using differential geometry concepts to study distances and curvatures. The Fisher scale is the primary tool in this field, used to measure the divergence between distributions, estimate the efficiency of estimators, and compare statistical models. Information engineering was founded by Amari (1985), who introduced the concept and studied families of exponential distributions. The field was later expanded to include the study of binary distributions (such as gamma, Mackay, Freund, and normal) through the work of Dodson (1999), and the studies by Abbas and Sabrie (2023) on the inverse gamma distribution using divergence measures such as the Kullback–Leibler divergence, J-divergence and geodesic equations (Peng & Sun, 2008). Amari (1985), Calin and Udriște (2014) also investigated the α -geometry of binary distributions. Abbas et al. (2025); Shanker et al. (2017) is considered a pioneer, having proposed Riemannian measures based on the Fisher Information Matrix, emphasizing their fundamental role in comparing distributions and understanding the complex relationships between statistical models. Our first step is to show that the S.M.L.D. distribution set represents a geometric structure (statistical manifold) that is extremely smooth and two-dimensional. Next, we derive the Riemann metric based on the estimation of the derivatives of the probability logarithm function (Shokeralla et al., 2025). We then proceed to calculate the Christoffel symbols, which enable us to deduce the geodesic equations representing the shortest paths on the junction. The work also includes an investigation of α connections and highlights the possibility of flattening the junction by employing a modified α connection. Furthermore, the distances between distributions are analyzed using Hellinger distance, with a special case examined where one of the parameters is fixed, which simplifies the model to a family of exponential distributions. This allows for the introduction of pairwise coordinates and the calculation of the associated canonical spacing (Bregman spacing).

The results of this study contribute to a deeper theoretical understanding of the geometric structure underlying the S.M.L.D. distribution. They provide innovative mathematical tools that can be effectively applied in multiple fields, such as

estimation theory, Bayesian inference, and machine learning, where geometric structures are used to improve algorithms and measure the similarity between statistical models (Mendes, 2024).

2. Preliminaries

Definition 2.1 (Calin & Udrişte, 2014): The family of probability density functions (p.d.f)

$M = \{f(x, \mu) : \mu = (\mu^1, \mu^2, \dots, \mu^n) \in E = R^+ \times R^+ \dots \times R^+\}$ is said to be a parametric model if there exists a mapping $h : E \rightarrow M$ which satisfies the following conditions:

- a) h is one-to-one
- b) The Wronskian determinant is non-zero.

Definition 2.2 (Shanker et al., 2017): Lindley's Distribution of Shanker and Mishra provides the p.d.f. for two parameters as follows:

$$f(x, \omega, \beta) = \frac{\beta^2}{\omega\beta + 1} (\omega + x)e^{-\beta x}, x > 0, \omega, \beta > 0$$

Definition 2.3 (Amari, 1982): The Christoffel symbols of the First Kind are given by

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

where g_{ij} are the components of the metric tensor and x^i, x^j, x^k These are the coordinates.

Definition 2.4 (Amari, 1985): The manifold with coordinate $\mu = (\mu^1, \mu^2, \dots, \mu^n)$ has geodesic equations provided by $\frac{d^2 \mu^k}{dt^2} + \sum \Gamma_{ijk} \frac{d\mu^i}{dt} \frac{d\mu^j}{dt} = 0$.

Definition 2.5 (Amari, 2016): Let $f_1(x)$ and $f_2(x)$ There are two points in the manifold M Then the Hellinger distance between f_1 and f_2 is given by

$$H^2(f_1, f_2) = 2 \int_0^\infty \left(\sqrt{f_1(x)} - \sqrt{f_2(x)} \right)^2 dx$$

3. Geometry of the S.M.L.D

Proposition (3.1): Let

$$M_L = \left\{ f(x, \omega, \beta) : f(x, \omega, \beta) = \frac{\beta^2}{\omega\beta + 1} (\omega + x)e^{-\beta x}, x > 0, (\omega, \beta) \in E \right\},$$

where $E = R^+ \times R^+$ Is the parameter space in two dimensions? Then M_L is a two-dimensional S.M.L.D manifold space parameterised by E .

Proof: The log-likelihood function

$$\ln(f(x)) = 2 \ln(\beta) - \ln(\omega\beta + 1) + \ln(\omega + x) - \beta x$$

has

$$\frac{\partial \ln f(x)}{\partial \omega} = \frac{1}{\omega + x} - \frac{\beta}{\omega\beta + 1}$$

$$\frac{\partial \ln f(x)}{\partial \beta} = \frac{2}{\beta} - \frac{\omega}{\omega\beta + 1} - x.$$

Then

$$\det \begin{bmatrix} \frac{1}{\omega + x} - \frac{\beta}{\omega\beta + 1} & \frac{2}{\beta} - \frac{\omega}{\omega\beta + 1} - x \\ -\frac{1}{(\omega + x)^2} + \frac{\beta^2}{(\omega\beta + 1)^2} & -\frac{2}{\beta^2} - \frac{\omega^2}{(\omega\beta + 1)^2} \end{bmatrix} \neq 0$$

Now, let $h: E \rightarrow M_L$ and $\ln f(x, \omega_1, \beta_1) = \ln f(x, \omega_2, \beta_2)$, for all $x > 0$

This implies

$$2 \ln(\beta_1) - \ln(\omega_1 \beta_1 + 1) + \ln(\omega_1 + x) - \beta_1 x = 2 \ln(\beta_2) - \ln(\omega_2 \beta_2 + 1) + \ln(\omega_2 + x) - \beta_2 x$$

$$\Rightarrow [2 \ln(\beta_1) - \ln(\omega_1 \beta_1 + 1)] - [2 \ln(\beta_2) - \ln(\omega_2 \beta_2 + 1)] + \ln\left(\frac{\omega_1 + x}{\omega_2 + x}\right) - (\beta_1 - \beta_2)x = 0$$

Let $A = 2 \ln(\beta_1) - \ln(\omega_1 \beta_1 + 1) - 2 \ln(\beta_2) + \ln(\omega_2 \beta_2 + 1)$, $B = \beta_1 - \beta_2$, and $g(x) = \ln\left(\frac{\omega_1 + x}{\omega_2 + x}\right) - Bx$. Then

$$g'(x) = \frac{\omega_2 - \omega_1}{(\omega_1 + x)(\omega_2 + x)} - B.$$

Since $g(x)$ is constant $\forall x > 0$, then $g'(x) = 0, \forall x > 0$,

so $\frac{\omega_2 - \omega_1}{(\omega_1 + x)(\omega_2 + x)} - B = 0, \forall x > 0$.

case 1: If $\omega_1 = \omega_2$, then $0 - B = 0 \rightarrow \beta_1 = \beta_2$, thus $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$.

case 2: $\omega_1 \neq \omega_2$, then $\frac{\omega_2 - \omega_1}{(\omega_1 + x)(\omega_2 + x)}$ varies with x , and cannot equal a constant B , for all $x > 0$ unless $\omega_2 - \omega_1 = 0$ which is a contradiction. Then the condition $\ln f(x, \omega_1, \beta_1) = \ln f(x, \omega_2, \beta_2)$, for all $x > 0$ necessitates $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$. So the map $h: E \rightarrow M_L$ is one-to-one indicating that the parameters of the Lindley distribution are uniquely determined by the logarithmic function.

So that M_L is a two-dimensional Lindley distribution manifold space parameterized by E .

Proposition (3.2): - $M_L = \{f(x, \omega, \beta): f(x, \omega, \beta) = \frac{\beta^2}{\omega\beta + 1}(\omega + x)e^{-\beta x}, x > 0, \omega, \beta > 0\}$, be the statistical manifold of the S.M.L.D. If $P \in M_L$ is a point corresponding to the distribution $f(x, \omega_0, \beta_0)$ and $V(P)$ be a neighborhood of the parameters (ω_0, β_0) . Then there exists a smooth function $F: M_L \rightarrow \mathbb{R}$ such that $F(\omega, \beta) = \begin{cases} 1 & \text{if } (\omega, \beta) = (\omega_0, \beta_0) \\ 0 & \text{if } (\omega, \beta) \notin V(P) \end{cases}$

Proof: Let $(\omega, \beta) \in \mathcal{P} = (0, \infty)^2$, and $\Omega: \mathcal{P} \rightarrow M_L$, such that $\Omega(\omega, \beta) = f(x, \omega, \beta)$. Since $\mathcal{P} = (0, \infty)^2$ is open in $\mathbb{R}^2$, then it is Hausdorff, second countable, and paracompact. So M_L is a smooth 2-dimensional manifold.

Since M_L is a smooth manifold, then there exists a chart (U, Φ) around P , where $\Phi: U \rightarrow \mathbb{R}^2$ is a diffeomorphism. Since $V(P)$ is a neighborhood of P , the intersection $U \cap V(P)$ is also neighborhood. Let r_1, r_2 be two radius such that $0 < r_1 < r_2$ and $\overline{B_{r_2}(\Phi(P))} \subset \Phi(U \cap V(P)) \subset \mathbb{R}^2$.

Let $W = \Phi^{-1}(B_{r_2}(\Phi(P)))$ be an open set satisfies $\overline{W} \subset U \cap V(P)$, and $\tilde{\eta}$ be a smooth one variable function defined as $\tilde{\eta} = \begin{cases} e^{-1/z} & z > 0 \\ 0 & z \leq 0 \end{cases}$,

$\mathcal{G}(t) = \frac{\tilde{\eta}(t)}{\tilde{\eta}(t) + \tilde{\eta}(1-t)}$. Now set $\Omega(\alpha) = 1 - \mathcal{G}\left(\frac{\alpha - r_1}{r_2 - r_1}\right), \alpha \geq 0$, define $\varsigma(u) = \mathcal{G}(\|u - \Phi(P)\|)$. Define $\tilde{F}(q) = \begin{cases} \varsigma(\Phi(q)), & q \in U \\ 0, & q \notin U \end{cases}$.

Therefore on U we have $\tilde{F} = \varsigma \circ \Phi$ is smooth function, while outside U is identically zero. Moreover $\text{supp } \tilde{F} \subset \Phi^{-1}(\text{supp } \varsigma) \subset \Phi^{-1}(B_{r_2}(\Phi(P))) = W \subset U$.

Then at P we have $F(P) = \varsigma(\Phi(P)) = 1$, outside $V(P)$ we have $F(q) = 0$ for any $q \notin V(P)$.

Proposition (3.3): - Let M_L be the 2-dimensional statistical manifold of the S.M.L.D with probability density function

$$f(x, \omega, \beta) = \frac{\beta^2}{\omega\beta + 1}(\omega + x)e^{-\beta x}, x > 0, \omega > 0$$

and let ∇^α be the α -Connection on M_L .

If M_L is curved with respect to the exponential connection ($\alpha=1$) then $\exists$ unique flattening coefficient $\alpha^* \in \mathbb{R}/\{1, -1\}$ such that M_L is flat with respect to the α^* -connection.

Proof: - The Riemann curvature tensor for the α -connection is given by

$$R_{ijk\ell}^{(\alpha)} = R_{ijk\ell}^{(1)} + \frac{1 - \alpha^2}{4}(T_{i\ell m}T_{jk}^m - T_{ikm}T_{j\ell}^m)$$

where $T_{ijk} = \partial_i \partial_j \partial_k \ell$ is the cubic tensor. Since M_L is curved with respect to $\nabla^{(1)}$, we have $R_{ijk\ell}^{(1)} \neq 0$ for some i, j, k, ℓ . The manifold M_L is flat with respect to $\nabla^{(\alpha^*)}$ if and only if $R_{ijk\ell}^{(\alpha^*)} = 0 \forall i, j, k, \ell \in \{1, 2\}$.

Substituting from the α -curvature decomposition

$$R_{ijk\ell}^{(1)} + \frac{1 - (\alpha^*)^2}{4}(T_{i\ell m}T_{jk}^m - T_{ikm}T_{j\ell}^m) = 0 \quad \dots (1)$$

This equation must be satisfied for all values of the indices i, j, k, ℓ

From the previous equation, we obtain,

$$\frac{1 - (\alpha^*)^2}{4} (T_{i\ell m} T_{jk}^m - T_{ikm} T_{j\ell}^m) = -R_{ijk\ell}^{(1)}$$

we observe that both sides represent fourth order tensors. For this equation to be consistently satisfied, there must be a proportional relationship between them

$$T_{i\ell m} T_{jk}^m - T_{ikm} T_{j\ell}^m = \lambda(\theta) R_{ijk\ell}^{(1)} \quad \dots \dots (2)$$

where $\lambda(\theta)$ is a scalar function depending on the parameters $\theta = (\omega, \beta)$ substituting from (2) into the original equation (1)

$$R_{ijk\ell}^{(1)} + \frac{1 - (\alpha^*)^2}{4} \lambda(\theta) R_{ijk\ell}^{(1)} = 0$$

$$\left[1 + \frac{1 - (\alpha^*)^2}{4} \lambda(\theta) \right] R_{ijk\ell}^{(1)} = 0$$

Since $R_{ijk\ell}^{(1)} \neq 0$, then

$$1 + \frac{1 - (\alpha^*)^2}{4} \lambda(\theta) = 0$$

$$\frac{1 - (\alpha^*)^2}{4} \lambda(\theta) = -1$$

$$1 - (\alpha^*)^2 = -\frac{4}{\lambda(\theta)}$$

$$(\alpha^*)^2 = 1 + \frac{4}{\lambda(\theta)} \quad \dots \dots \dots (3)$$

To ensure that α^* is a real number we must have: $1 + \frac{4}{\lambda(\theta)} \geq 0$

And since $\alpha^* \neq \mp 1$, then $1 + \frac{4}{\lambda(\theta)} > 0$

$$\text{Therefore } \alpha^* = \sqrt{1 + \frac{4}{\lambda(\theta)}}$$

This proves the existence of a unique flattening coefficient. α^* that transforms the Shanker Mishra Lindley manifold into a flat manifold with respect to α -Connection

Proposition (3.4): - let M_L be the 2-dimensional S.M.L.D manifold with the coordinate system (ω, β) . Then the distance between two distributions is given by

$$ds^2 = \left[\frac{\beta^2}{\omega\beta + 1} e^{\omega\beta} E_1(\omega\beta) \right] (d\omega)^2 + \frac{2}{(\omega\beta + 1)^2} d\omega d\beta + \left[\frac{2}{\beta^2} - \frac{\omega^2}{(\omega\beta + 1)^2} \right] (d\beta)^2 \quad \dots (3)$$

Proof: -

The log-likelihood function of the S.M.L.D is given by

$$\ell = \ln(f(x)) = 2 \ln(\beta) - \ln(\omega\beta + 1) + \ln(\omega + x) - \beta x$$

So,

$$\partial_\omega \ln(f, x, \omega, \beta) = \frac{1}{\omega + x} - \frac{\beta}{\omega\beta + 1}$$

$$\partial_\beta \ln(f, x, \omega, \beta) = \frac{2}{\beta} - \frac{\omega}{\omega\beta + 1} - x$$

Let the Riemannian metric be

$$g_{ij} = 4 \int_0^\infty \frac{\partial}{\partial \lambda^i} (\sqrt{f(x, \lambda)}) \cdot \frac{\partial}{\partial \lambda^j} (\sqrt{f(x, \lambda)}) dx, i, j = 1, 2,$$

where $\lambda = (\lambda^1, \lambda^2) = (\omega, \beta)$, $x > 0$ and $\omega, \beta > 0$

$g_{11} = \frac{\beta^2}{\omega\beta + 1} e^{\omega\beta} E_1(\omega\beta)$, such that $E_1(\omega\beta)$ is the exponential integral function

$$g_{22} = \frac{2}{\beta^2} - \frac{\omega^2}{(\omega\beta + 1)^2}, g_{21} = g_{12} = \frac{1}{(\omega\beta + 1)^2}$$

Since

$$ds^2 = g_{11}(d\omega)^2 + g_{12}d\omega d\beta + g_{21}d\beta d\omega + g_{22}(d\beta)^2$$

then the distance between two distributions is given by (3).

Preposition (3.5): The geodesic equation of the S.M.L.D of two parameters are given by (4) and (5).

Proof: - by definition (2.3) we find

$$\begin{aligned}\Gamma_{111} &= \frac{\beta^3}{2} \left(\frac{1}{\omega\beta + 1} - \frac{1}{(\omega\beta + 1)^2} \right) e^{\omega\beta} E_1(\omega\beta) - \frac{\beta^2}{\omega(\omega\beta + 1)} \\ \Gamma_{112} &= \frac{-2\beta}{(\omega\beta + 1)^3} - \frac{\beta}{\omega\beta + 1} e^{\omega\beta} E_1(\omega\beta) + \frac{\omega\beta^2}{2(\omega\beta + 1)^2} e^{\omega\beta} E_1(\omega\beta) + \frac{\beta^2}{2(\omega\beta + 1)} \\ \Gamma_{211} &= \Gamma_{121} = \frac{\beta}{\omega\beta + 1} e^{\omega\beta} E_1(\omega\beta) - \frac{\omega\beta^2}{2(\omega\beta + 1)^2} e^{\omega\beta} E_1(\omega\beta) \\ \Gamma_{221} &= \Gamma_{212} = \frac{-\omega\beta^2}{(\omega\beta + 1)^3} \\ \Gamma_{222} &= \frac{-2}{\beta^3} + \frac{\omega^3}{(\omega\beta + 1)^3} \\ \Gamma_{122} &= \frac{\omega}{(\omega\beta + 1)^3} + \frac{\omega^2}{(\omega\beta + 1)^3}\end{aligned}$$

Then the geodesic equation for ω is given by (4)

$$\begin{aligned}& \frac{d^2\omega}{dt^2} + \left(\frac{\beta^3}{2} \left(\frac{1}{\omega\beta + 1} - \frac{1}{(\omega\beta + 1)^2} \right) e^{\omega\beta} E_1(\omega\beta) - \frac{\beta^2}{\omega(\omega\beta + 1)} \right) \left(\frac{d\omega}{dt} \right)^2 \\ & + 2 \left(\frac{\beta}{\omega\beta + 1} e^{\omega\beta} E_1(\omega\beta) - \frac{\omega\beta^2}{2(\omega\beta + 1)^2} e^{\omega\beta} E_1(\omega\beta) \right) \frac{d\omega}{dt} \frac{d\beta}{dt} + \frac{-\omega\beta^2}{(\omega\beta + 1)^3} \left(\frac{d\beta}{dt} \right)^2 = 0\end{aligned} \quad \dots (4)$$

The geodesic equation for β is (5)

$$\begin{aligned}& \frac{d^2\beta}{dt^2} + \left[\frac{-2\beta}{(\omega\beta + 1)^3} - \frac{\beta}{\omega\beta + 1} e^{\omega\beta} E_1(\omega\beta) + \frac{\omega\beta^2}{2(\omega\beta + 1)^2} e^{\omega\beta} E_1(\omega\beta) + \frac{\beta^2}{2(\omega\beta + 1)} \right] \left(\frac{d\omega}{dt} \right)^2 \\ & + 2 \left[\frac{\omega}{(\omega\beta + 1)^3} + \frac{\omega^2}{(\omega\beta + 1)^3} \right] \frac{d\omega}{dt} \frac{d\beta}{dt} + \left[\frac{-2}{\beta^3} + \frac{\omega^3}{(\omega\beta + 1)^3} \right] \left(\frac{d\beta}{dt} \right)^2 = 0\end{aligned} \quad \dots (5)$$

Proposition (3.6): Let $f_1(x)$ and $f_2(x)$ are two points in the parametrized surface of the S.M.L.D then the Hellinger distance between f_1 and f_2 is given by

$$H(f_1: f_2) = \sqrt{2} \sqrt{C_1^2 I_1 + C_2^2 I_2 + 2C_1 C_2 I_3}$$

Proof: - The Hellinger distance is

$$H^2(f_1, f_2) = 2 \int_0^\infty \left(\sqrt{f_1(x)} - \sqrt{f_2(x)} \right)^2 dx$$

Then

$$\begin{aligned}H(f_1: f_2) &= 2 \int_0^\infty \left[\left(\sqrt{\frac{\beta_1^2}{\omega_1\beta_1 + 1} (\omega_1 + x) e^{-\beta_1 x}} - \sqrt{\frac{\beta_2^2}{\omega_2\beta_2 + 1} (\omega_2 + x) e^{-\beta_2 x}} \right) \right]^2 dx \\ \sqrt{f_1(x)} &= \sqrt{\frac{\beta_1^2}{\omega_1\beta_1 + 1}} \sqrt{(\omega_1 + x) e^{-\frac{\beta_1}{2} x}}, \sqrt{f_2(x)} = \sqrt{\frac{\beta_2^2}{\omega_2\beta_2 + 1}} \sqrt{(\omega_2 + x) e^{-\frac{\beta_2}{2} x}}\end{aligned}$$

Let $C_1 = \sqrt{\frac{\beta_1^2}{\omega_1\beta_1+1}}$ and $C_2 = \sqrt{\frac{\beta_2^2}{\omega_2\beta_2+1}}$. Then

$$H(f_1: f_2) = 2 \int_0^\infty \left(\sqrt{f_1(x)} - \sqrt{f_2(x)} \right)^2 dx = \int_0^\infty \left(C_1 \sqrt{(\omega_1+x)} e^{-\frac{\beta_1}{2}x} - C_2 \sqrt{(\omega_2+x)} e^{-\frac{\beta_2}{2}x} \right)^2 dx$$

Let

$$I_1 = \int_0^\infty \sqrt{(\omega_1+x)} e^{-\frac{\beta_1}{2}x} dx = \frac{\omega_1}{\beta_1} + \frac{1}{\beta_1^2}$$

$$I_2 = \int_0^\infty \sqrt{(\omega_2+x)} e^{-\frac{\beta_2}{2}x} dx = \frac{\omega_2}{\beta_2} + \frac{1}{\beta_2^2}$$

$$I_3 = \int_0^\infty \sqrt{(\omega_1+x)(\omega_2+x)} e^{-\frac{(\beta_1+\beta_2)}{2}x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{(\omega_1+\omega_2+\sqrt{\omega_1\omega_2})}{\left(\frac{\beta_1-\beta_2}{2}\right)} \right).$$

So,

$$H(f_1: f_2) = \sqrt{2} \left[\left(\frac{\beta_1^2}{\omega_1\beta_1+1} \right) \left(\frac{\omega_1}{\beta_1} + \frac{1}{\beta_1^2} \right) + \left(\frac{\beta_2^2}{\omega_2\beta_2+1} \right) \left(\frac{\omega_2}{\beta_2} + \frac{1}{\beta_2^2} \right) - 2 \left(\sqrt{\frac{\beta_1^2\beta_2^2}{(\omega_1\beta_1+1)(\omega_2\beta_2+1)}} \right) I_3 \right]$$

4. Sub Manifold: $M_{L^*} \subset M_L : \omega$ is Fixed in S.M.L.D

If ω is fixed in S.M.L.D then the S.M.L.D reduce to the form: $f(x, \beta) = \frac{\beta^2}{\omega\beta+1} (\omega+x) e^{-\beta x}$, $x > 0, \beta > 0$, observing that the sub manifold M_{L^*} defines an exponential family, so we proceed to determined its dual coordinates, Geodesic Equation and divergence between two points as follows:

Position 4.1): Let cap M sub, cap L to the asterisk operator, end subscript be a one-dimensional sub manifold of a statistical manifold, subscript base, cap M, end base, sub cap L, Position 4.1): Let M_{L^*} be one-dimensional sub manifold of statistical manifold M_L which forms an exponential family. The dual coordinates (ϑ, ρ) on M_{L^*} are given by $\vartheta = -\beta$, $\rho = \frac{2+\omega\beta}{-\beta(1+\omega\beta)}$, where (ϑ, ρ) constitutes a Legendre dual pair satisfying $\frac{d\rho}{d\vartheta} > 0$. The Fisher metric on M_{L^*} is

$$g = \frac{-2 - 4\omega\beta - \omega^2\beta^2}{\beta^2(1+\omega\beta)^2}$$

Proof: - The p.d.f of S.M.L.D for fixed ω is

$$f(x, \beta) = \frac{\beta^2}{\omega\beta+1} (\omega+x) e^{-\beta x}.$$

Can be expressed in exponential family form:

$$\begin{aligned} f(x, \beta) &= \exp[\ln(\omega+x) + 2 \ln \beta - \ln(1+\omega\beta) - (\beta x)] \\ &= \exp[-\beta x + \ln(\omega+x) + 2 \ln \beta - \ln(1+\omega\beta)] \end{aligned}$$

Let $\vartheta = -\beta$, then

$$f(x, \vartheta) = \exp[\vartheta x + \ln(\omega+x) + 2 \ln(-\vartheta) - \ln(1-\omega\vartheta) - \Omega(\vartheta)],$$

where the cumulate function is:

$$\Omega(\vartheta) = 2 \ln(-\vartheta) - \ln(1-\omega\vartheta).$$

Thus, M_{L^*} forms a one-parameter exponential family with natural parameter $\vartheta = -\beta$. The mean parameter is given by

$$\rho = E[T(X)] = \Omega'(\vartheta) = \frac{2}{\vartheta} + \frac{\omega}{1-\omega\vartheta}$$

$$\begin{aligned}
\text{Substituting } \vartheta = -\beta, \rho = \Omega'(-\beta) &= -\frac{2}{\beta} + \frac{\omega}{1+\omega\beta} \\
&= \frac{-2(1+\omega\beta) + \omega\beta}{\beta(1+\omega\beta)} \\
&= \frac{-2-2\omega\beta + \omega\beta}{\beta(1+\omega\beta)} \\
&= \frac{-2-2\omega\beta}{\beta(1+\omega\beta)} = \frac{2+\omega\beta}{-\beta(1+\omega\beta)}
\end{aligned}$$

The Legendre duality requires that $\rho = \Omega'(\vartheta)$ and $\vartheta = \Phi'(\rho)$, where Φ is the Legendre transform of $\Omega : \Phi(\rho) = \vartheta \rho - \Omega(\vartheta)$. The positive definiteness condition: $\frac{d\rho}{d\vartheta} = \Omega''(\vartheta) > 0$

$$\Omega''(\vartheta) = -\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}$$

Substituting, $\vartheta = -\beta$

$$\begin{aligned}
\Omega''(-\beta) &= \frac{-2}{\beta^2} + \frac{\omega^2}{(1+\omega\beta)^2} \\
&= \frac{-2(1+\omega\beta)^2 + \omega^2\beta^2}{\beta^2(1+\omega\beta)^2} \\
&= \frac{-2(1+2\omega\beta + \omega^2\beta^2) + \omega^2\beta^2}{\beta^2(1+\omega\beta)^2} \\
&= \frac{-2-4\omega\beta - \omega^2\beta^2}{\beta^2(1+\omega\beta)^2} = g_{ij}
\end{aligned}$$

Since $\omega, \beta > 0$ The denominator is positive and the numerator is negative, so the overall expression is positive. Therefore $\frac{d\rho}{d\vartheta} = \Omega''(\vartheta) > 0$

The expressions are well-defined for $\beta > 0$ and $1 + \omega\beta \neq 0$.

Proposition (4.2): Geodesic Equation for S.M.L.D statistical manifold M_{L^*} with dual Coordinates (ϑ, ρ) is

$$\frac{d^2\vartheta}{dt^2} + \frac{\frac{2}{\vartheta^3} + \frac{\omega^3}{(1-\omega\vartheta)^3}}{-\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}} \left(\frac{d\vartheta}{dt}\right)^2 = 0$$

And

$$\frac{d^2\rho}{dt^2} - \frac{1}{2} \left(\frac{\frac{6}{\vartheta^4} + \frac{6\omega^4}{(1-\omega\vartheta)^4}}{\left(-\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}\right)^2} \right) \left(\frac{d\rho}{dt}\right)^2 = 0$$

Proof: -

The Riemannian metric is given by:

$$\begin{aligned}
g_{ij} &= \frac{\partial^2 \Omega}{\partial \vartheta^i \partial \vartheta^j}, \quad g^{ij} = \frac{\partial^2 \Phi}{\partial \rho_i \partial \rho_j} \\
g_{\vartheta\vartheta} &= -\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2} = \Omega''(\vartheta) \\
g_{\rho\rho} &= \frac{1}{-\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}} = \Phi''(\rho) = \frac{1}{\Omega''(\vartheta)}
\end{aligned}$$

where $\Phi(\rho)$ is the Legendre transform of $\Omega(\vartheta)$

For a one-dimensional manifold, the geodesic equation in coordinate ϑ is

$$\frac{d^2\vartheta}{dt^2} + \Gamma_{\vartheta\vartheta}^{\vartheta} \left(\frac{d\vartheta}{dt} \right)^2 = 0$$

$$\Gamma_{\vartheta\vartheta}^{\vartheta} = \frac{1}{2} g^{\vartheta\vartheta} \frac{\partial g_{\vartheta\vartheta}}{\partial \vartheta} = \frac{1}{2} \cdot \frac{1}{g_{\vartheta\vartheta}} \cdot \frac{\partial g_{\vartheta\vartheta}}{\partial \vartheta} = \frac{1}{2} \left(\frac{1}{\frac{-2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}} \right) \left(\frac{4}{\vartheta^3} + \frac{2\omega^3}{(1-\omega\vartheta)^3} \right)$$

Then the geodesic equation in ϑ is

$$\frac{d^2\vartheta}{dt^2} + \frac{\frac{2}{\vartheta^3} + \frac{\omega^3}{(1-\omega\vartheta)^3}}{\frac{-2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}} \left(\frac{d\vartheta}{dt} \right)^2 = 0$$

While the geodesic equation in ρ is

$$\frac{d^2\rho}{dt^2} + \Gamma_{\rho\rho}^{\rho} \left(\frac{d\rho}{dt} \right)^2 = 0$$

$$\Gamma_{\rho\rho}^{\rho} = \frac{1}{2} g^{\rho\rho} \frac{\partial g_{\rho\rho}}{\partial \rho}$$

Using $g_{\rho\rho} = \Phi''(\rho)$ and $\Phi''(\rho) = \frac{1}{\Omega''(\vartheta)}$

Then $\Gamma_{\rho\rho}^{\rho} = -\frac{1}{2} \Omega''(\vartheta) \frac{\partial}{\partial \rho} \left(\frac{1}{\Omega''(\vartheta)} \right)$, Since $\Omega(\vartheta) = \ln \left(\frac{1-\omega\vartheta}{\vartheta^2} \right) + C$, then

$$\Omega'(\vartheta) = \frac{2}{\vartheta} + \frac{\omega}{1-\omega\vartheta}, \quad \Omega''(\vartheta) = \frac{-2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2} \text{ and}$$

$$g_{\rho\rho} = \frac{1}{\Omega''(\vartheta)} = \frac{1}{-\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}}$$

$$\Gamma_{\rho\rho}^{\rho} = \frac{1}{2} g^{\rho\rho} \frac{\partial g_{\rho\rho}}{\partial \rho} = \frac{1}{2} \Omega''(\vartheta) \frac{\partial}{\partial \rho} \left(\frac{1}{\Omega''(\vartheta)} \right)$$

Since $\frac{\partial}{\partial \rho} \left(\frac{1}{\Omega''(\vartheta)} \right) = \frac{1}{\Omega''(\vartheta)} \frac{\partial}{\partial \vartheta} \left(\frac{1}{\Omega''(\vartheta)} \right)$ and $\frac{\partial}{\partial \vartheta} \left(\frac{1}{\Omega''(\vartheta)} \right) = -\frac{\Omega'''(\vartheta)}{[\Omega''(\vartheta)]^2}$

Where $\Omega'''(\vartheta) = \frac{6}{\vartheta^4} + \frac{6\omega^4}{(1-\omega\vartheta)^4}$. So

$$\Gamma_{\rho\rho}^{\rho} = \frac{1}{2} \Omega''(\vartheta) \frac{1}{\Omega''(\vartheta)} \left[\frac{-\Omega'''(\vartheta)}{[\Omega''(\vartheta)]^2} \right]$$

$$= -\frac{1}{2} \frac{\Omega'''(\vartheta)}{[\Omega''(\vartheta)]^2}$$

Then the geodesic equation ρ is

$$\frac{d^2\rho}{dt^2} + \Gamma_{\rho\rho}^{\rho} \left(\frac{d\rho}{dt} \right)^2 = 0$$

$$\frac{d^2\rho}{dt^2} - \frac{1}{2} \frac{\Omega'''(\vartheta)}{[\Omega''(\vartheta)]^2} \left(\frac{d\rho}{dt} \right)^2 = 0$$

$$\frac{d^2\rho}{dt^2} - \frac{1}{2} \left(\frac{\frac{6}{\vartheta^4} + \frac{6\omega^4}{(1-\omega\vartheta)^4}}{\left(\frac{-2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2} \right)^2} \right) \left(\frac{d\rho}{dt} \right)^2 = 0$$

Proposition (4.3): Let M_{L^*} be the one-dimensional exponential family derived from S.M.L.D by fixing the parameter ω , with dual coordinates (ϑ, ρ) . The canonical divergence associated with the dual coordinates is expressed as

$$D[P(x, \vartheta_1): Q(x, \vartheta_2)] = \Omega(\vartheta_2) - \Omega(\vartheta_1) - \rho_1(\vartheta_2 - \vartheta_1)$$

Where

$$\rho_1 = \Omega'(\vartheta_1) \text{ and } \Omega(\vartheta) = \ln\left(\frac{|1 - \omega\vartheta|}{\vartheta^2}\right) + c$$

is the cumulative function of the exponential family.

Equivalently, in terms of the mean coordinate ρ :

$$D[P(x, \rho_1): Q(x, \rho_2)] = \Phi(\rho_1) - \Phi(\rho_2) - \vartheta_2(\rho_1 - \rho_2)$$

where $\Phi(\rho) = \vartheta\rho - \Omega(\vartheta)$

is the Legendre transform of Ω . The divergence D is non-negative and vanishes if and only if $\vartheta_1 = \vartheta_2$ (or equivalently, $\rho_1 = \rho_2$)

Proof:

let P and Q There are two points on M_L^* . For a dually flat statistical manifold with dual coordinates (θ, ρ) and potential function $\Omega(\vartheta)$ and $\Phi(\rho)$ related by Legendre transformation

$$\phi(\rho) = \vartheta\rho - \Omega(\vartheta), \rho = \Omega'(\vartheta), \vartheta = \Phi'(\rho)$$

The canonical divergence between two points P and Q is given by

$$D[P: Q] = \Omega(\vartheta_P) + \Phi(\rho_Q) - \vartheta_P\rho_Q,$$

Or equivalently

$$D[P: Q] = \Omega(\vartheta_Q) - \Omega(\vartheta_P) - \rho_P(\vartheta_Q - \vartheta_P)$$

The Divergence in ϑ -Coordinates are

$$D[P(x, \vartheta_1): Q(x, \vartheta_2)] = \Omega(\vartheta_2) - \Omega(\vartheta_1) - \rho_1(\vartheta_2 - \vartheta_1)$$

Substitute $\rho_1 = \Omega'(\vartheta_1)$ we have

$$D[P(x, \vartheta_1): Q(x, \vartheta_2)] = \Omega(\vartheta_2) - \Omega(\vartheta_1) - \Omega'(\vartheta_1)(\vartheta_2 - \vartheta_1) \quad \dots \dots (1)$$

The Divergence in ρ - coordinates

$$D[P(x, \rho_1): Q(x, \rho_2)] = \Phi(\rho_1) - \Phi(\rho_2) - \vartheta_2(\rho_1 - \rho_2)$$

Where $\vartheta_2 = \Phi'(\rho_2)$

For any strictly convex function Ω , the Bregman divergence satisfies:

$D[P: Q] \geq 0$, with equality if and only if $\vartheta_1 = \vartheta_2$ or $\rho_1 = \rho_2$

since $\Omega''(\vartheta) = -\frac{2}{\vartheta^2} + \frac{\omega^2}{(1-\omega\vartheta)^2}$, and $\Omega''(\vartheta) > 0$ for all $\vartheta = -\beta$ with $\beta > 0$ then Ω is strictly convex.

Substituting the specific Ω For S.M.L.D in (1), we have

$$D[P: Q] = \ln\left(\frac{|1 - \omega\vartheta_2|}{\vartheta_2^2}\right) - \ln\left(\frac{|1 - \omega\vartheta_1|}{\vartheta_1^2}\right) - \left(\frac{2}{\vartheta_1} - \frac{\omega}{1 - \omega\vartheta_1}\right)(\vartheta_2 - \vartheta_1)$$

In terms of β

$$D[P: Q] = \ln\left(\frac{1 + \omega\beta_2}{\beta_2^2}\right) - \ln\left(\frac{1 + \omega\beta_1}{\beta_1^2}\right) - \left(\frac{-2}{\beta_1} + \frac{\omega}{1 + \omega\beta_1}\right)(-\beta_2 + \beta_1)$$

5. Conclusions

The study concluded that the two-parameter Lindley distribution forms a smooth two-dimensional statistical manifold that can be precisely represented geometrically within the parameter space (ω, β) . The derived Riemannian metric, based on the Fisher information matrix, enables the characterisation of distances and curvatures between distributions and reflects the structural relationship among the model's parameters. The statistical manifold is found to be curved under exponential continuity. ($\alpha = 1$). It is evident that flattening the manifold is achieved only when a specific value for the flattening factor is chosen that satisfies the required condition. Geodesic equations and Hellinger distance were also derived to measure the convergence of Lindley distributions. The results demonstrate that applying computer geometry to non-exponential distributions strengthens the theoretical foundations of statistical geometry and supports the development of more precise methods for parameter estimation and model analysis. This work contributes to the development of a unified mathematical framework that enables the study of high-dimensional distributions and deepens the geometric understanding of their

properties.

6. References

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