

Fréchet Current Records: Bayesian and Non-Bayesian Estimation and Prediction with Application to Saudi Industrial Data

Ramy Abdelhamid Aldallal

Department of Management, College of Business Administration Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Saudi Arabia

ABSTRACT

This paper examines estimation and prediction problems based on current-record statistics derived from the Fréchet distribution (the inverse Weibull distribution). The Fréchet model is widely used in reliability engineering, extreme-value analysis, and environmental studies, particularly for modeling phenomena with heavy-tailed distributions and extreme events. Assuming that the observed data constitute current records from a two-parameter Fréchet distribution with shape parameter α and scale parameter s , we derive the corresponding probability density function (PDF) and cumulative distribution function (CDF) for both upper and lower current records. Closed-form expressions for the moments of these record values are then obtained. Subsequently, maximum-likelihood and Bayesian estimate of the model parameters are developed, along with procedures for predicting future current record values and constructing predictive intervals. A Monte Carlo simulation study is conducted to evaluate the performance of the proposed estimators under different sample sizes and parameter configurations. Finally, the proposed methodology is illustrated through an application to industrial data from Saudi Arabia, highlighting the practical importance of Fréchet record modeling in reliability and life-testing analyses.

ARTICLE HISTORY

Received January 3, 2026

Accepted February 10, 2026

KEYWORDS

Fréchet Distribution,
Current Records,
Bayesian and Non-Bayesian
Estimation and Prediction,
Moments,
Saudi Arabian Industry Data

1. Introduction

Record values arise naturally in many disciplines such as industrial reliability testing, climatology, sports analytics, and finance. A record is observed whenever an outcome exceeds all previous observations (upper record) or falls below them (lower record). In numerous practical situations, only the most recent record or a sequence of recent records is available for analysis; such observations are referred to as current records.

The Fréchet distribution, also known as the inverse Weibull distribution, is a fundamental model in extreme value theory and is particularly suitable for modeling heavy-tailed lifetime and environmental data. Owing to its ability to capture extreme behavior, it has been extensively employed in reliability analysis and risk assessment studies. When observations are available as record values rather than complete samples, standard inferential procedures are no longer applicable, as the likelihood function must be constructed from record-based distributions. Consequently, specialised estimation and prediction techniques are required.

Predicting future current record values is of significant importance in a wide range of statistical, engineering, and scientific applications. Current records, whether upper, lower, or based on their ranges, represent the most extreme realizations within a sequential process and therefore convey critical information about structural changes in the underlying system. Accurate forecasting of future record values enhances risk evaluation, supports informed decision-making, and provides early warnings of potential failures or extreme events in fields such as reliability engineering, climate science, finance, and environmental monitoring.

This study develops a comprehensive inferential framework for current record data modeled by the two-parameter Fréchet distribution. Classical and Bayesian estimation methods are presented, predictive intervals are derived, and the proposed methodology is applied to industrial data from Saudi Arabia. The study is particularly relevant to the Kingdom of Saudi Arabia's Vision 2030 initiative, which emphasizes leadership in energy and industry as a strategic national objective. Reliable prediction of record-based performance measures directly improves the robustness, efficiency, and durability of

critical industrial and energy systems, thereby supporting economic diversification and technological progress.

This work builds upon an extensive body of literature on current record theory. Aldallal (2026) used Bayesian estimation and prediction approaches for Weibull current records and estimated the parameters of the Rayleigh current record distribution with prediction of future intervals. Barakat et al. (2014) established prediction intervals for current records under any continuous distribution, while Raqab (2009) developed distribution-free prediction intervals for future current records. Ahmadi and Balakrishnan (2005) constructed confidence intervals for quantile intervals based on current records, an approach later extended to k-records by Chahkandi and Ahmadi (2013).

Additional contributions have further enriched the field. Muraleedharan and Chacko (2025) examined prediction of future k-record values, while Alsanea (2024) provided a comprehensive review of the role of record values in statistical inference. Arnold and Press (1989) developed Bayesian estimation and prediction methods for Pareto data, and Basak and Balakrishnan (2003) investigated likelihood-based prediction of future record statistics. Chacko and Mary (2013) studied estimation and prediction using k-record values from the normal distribution, whereas Einmahl and Magnus (2008) analyzed world record data from an applied statistical perspective. Empacher and Kamps (2025) contributed to simultaneous prediction of record values. Foundational Bayesian methodology was detailed by Gelman et al. (2013), while Gulati and Padgett (2003) discussed inference from record-breaking data. More recent developments include works by Kim and Seo (2020), AL-Dayian et al. (2024), El-Din et al. (2015), Sana and Faizan (2021), Shafay (2022), and Volovski and Kamps (2020), all of which contribute to modern prediction and estimation techniques for record values.

In this paper, we adopt the following notation for a sequence $\{X_\mu\}$. Let L_μ^{c*} and U_μ^{c*} denote the lower and upper current records, respectively, observed at the m th record occurrence. By definition, $L_{\mu+1}^{c*} = L_\mu^{c*}$ when a new upper record is observed ($U_\mu^{c*} > U_m^{c*}$), and $U_\mu^{c*} = U_\mu^{c*}$ when a new lower record occurs ($L_{\mu+1}^{c*} < L_\mu^{c*}$). Hence, L_μ^{c*} represents the minimum observation up to the μ th record time, with initial values given by $U_0^{c*} = L_0^{c*} = X_1$. For $\mu \geq 1$, the interval (L_μ^{c*}, U_μ^{c*}) is referred to as the record coverage, and the current record range is defined as $R_\mu^{c*} = U_\mu^{c*} - L_\mu^{c*}$. A new record range is registered whenever either a new upper or a new lower current record is observed.

Section 2 derives the cumulative distribution function (CDF) and probability density function (PDF) for both upper and lower current records when the underlying data follows a two-parameter Fréchet distribution with shape parameter α and scale parameter s . Section 3 presents closed-form expressions for the moments of these records, enabling analysis of their central tendency and dispersion. Parameter estimation procedures, including both maximum likelihood and Bayesian methods for upper and lower current records, are discussed in Section 4. Section 5 focuses on predicting future upper and lower current records and record ranges and develops corresponding prediction intervals using two approaches. Finally, Section 6 consists of two parts: a Monte Carlo simulation study assessing the performance of the proposed estimators and predictors, and an application to Saudi Arabian industrial data illustrating the practical effectiveness of the proposed methodology.

2. CDF and PDF of Fréchet Current Records

Determining the CDF and PDF of a dataset is a fundamental step in statistical analysis, as these functions provide a complete probabilistic description of the data's distribution. This information is crucial not only for accurate parameter estimation but also for reliable predictions, risk assessment, and decision-making in applied fields such as finance, engineering, environmental studies, and reliability testing. In particular, the study of current record values—observations that exceed all previous measurements (upper current records) or fall below them (lower current records)—is essential for understanding the dynamics of rare or noteworthy events in a sequence of data. In this section, we derive the CDF and PDF of the current lower and upper records when the original dataset follows a Fréchet distribution. These derivations form the basis for current record analysis, providing valuable insights for statistical inference, probabilistic modeling, and practical applications where monitoring record-breaking events is of interest. The PDF and CDF of the Fréchet distribution with scale parameter $s > 0$ and shape parameter $\alpha > 0$ are given respectively by:

$$f(x; \alpha, s) = \frac{\alpha}{s} \left(\frac{x}{s}\right)^{-(\alpha+1)} \exp \left[-\left(\frac{x}{s}\right)^{-\alpha} \right], x > 0, \quad (1)$$

$$F(x; \alpha, s) = \exp \left[-\left(\frac{x}{s}\right)^{-\alpha} \right], x > 0. \quad (2)$$

The parameter α controls the tail behavior of the Fréchet distribution:

- For $0 < \alpha < 1$, the distribution exhibits an extremely heavy right tail with infinite mean and variance.
- For $1 \leq \alpha < 2$, the mean exists but the variance is infinite.
- For $\alpha \geq 2$, both the mean and variance are finite.
- As α increases, the distribution becomes less heavy-tailed and more concentrated around its central values.

Theorem 1. For $\mu > 1$, the CDF and PDF of the lower (upper) current records, when the original data follows a

Fréchet distribution with shape parameter $\alpha > 0$ and scale parameter $s > 0$, are respectively given by:

$$F_{L_{\mu}^{c*}}(x; \alpha, s) = 1 + 2^{\mu} \exp \left[-\left(\frac{x}{s}\right)^{-\alpha} \right] P \left(\mu, \left(\frac{x}{s}\right)^{-\alpha} \right) - P \left(\mu, 2 \left(\frac{x}{s}\right)^{-\alpha} \right), \quad (3)$$

$$f_{L_{\mu}^{c*}}(x; \alpha, s) = 2^{\mu} \frac{\alpha}{s} \left(\frac{x}{s}\right)^{-(\alpha+1)} \exp \left[-\left(\frac{x}{s}\right)^{-\alpha} \right] P \left(\mu, \left(\frac{x}{s}\right)^{-\alpha} \right), \quad (4)$$

$$F_{U_{\mu}^{c*}}(x; \alpha, s) = P(\mu, -2 \log(1 - \exp[-(x/s)^{-\alpha}])) - 2^{\mu} (1 - \exp[-(x/s)^{-\alpha}]) \times P(\mu, -\log(1 - \exp[-(x/s)^{-\alpha}])) \quad (5)$$

$$f_{U_{\mu}^{c*}}(x; \alpha, s) = 2^{\mu} \frac{\alpha}{s} \left(\frac{x}{s}\right)^{-(\alpha+1)} \exp[-(x/s)^{-\alpha}] P(\mu, -\log(1 - \exp[-(x/s)^{-\alpha}])) \quad (6)$$

Where $P(a, b)$ denotes the regularized lower Gamma function.

Proof. To find the CDF and PDF of the current records when the data follows a Fréchet distribution, we begin with the general Houchens formulas (Houchens, 1984):

$$F_{L_{\mu}^{c*}}(x) = 2^{\mu} F(x) \left\{ 1 - F(x) \sum_{k=0}^{\mu-1} 2^{-(k+1)} \sum_{j=0}^k \frac{[-2 \log F(x)]^j}{j!} \right\}, \quad (7)$$

$$f_{L_{\mu}^{c*}}(x) = 2^{\mu} f(x) \left\{ 1 - F(x) \sum_{k=0}^{\mu-1} \frac{[-\log F(x)]^k}{k!} \right\}, \quad (8)$$

$$F_{U_{\mu}^{c*}}(x) = 1 - 2^{\mu} \bar{F}(x) \left\{ 1 - \bar{F}(x) \sum_{k=0}^{\mu-1} 2^{-(k+1)} \sum_{j=0}^k \frac{[-2 \log \bar{F}(x)]^j}{j!} \right\}, \quad (9)$$

$$f_{U_{\mu}^{c*}}(x) = 2^{\mu} f(x) \left\{ 1 - \bar{F}(x) \sum_{k=0}^{\mu-1} \frac{[-\log \bar{F}(x)]^k}{k!} \right\}, \quad (10)$$

where $\bar{F}(x) = 1 - F(x)$.

Substitute (1) and (2) expressions into the formula for (7) and define

$$t = \left(\frac{x}{s}\right)^{-\alpha}$$

Then

$$F_{L_{\mu}^{c*}}(x) = 2^{\mu} e^{-t} \left[1 - e^{-t} \sum_{k=0}^{\mu-1} 2^{-(k+1)} \sum_{j=0}^k \frac{(2t)^j}{j!} \right].$$

Let the inner double sum be

$$S = \sum_{k=0}^{\mu-1} 2^{-(k+1)} \sum_{j=0}^k \frac{(2t)^j}{j!}.$$

Changing the order of summation yields

$$S = \sum_{j=0}^{\mu-1} \frac{t^j}{j!} - 2^{-\mu} \sum_{j=0}^{\mu-1} \frac{(2t)^j}{j!}.$$

Using the identity

$$\sum_{r=0}^{\mu-1} \frac{z^r}{r!} = e^z \left(1 - \frac{\gamma(\mu, z)}{(\mu-1)!} \right),$$

we obtain

$$S = e^t \left(1 - \frac{\gamma(\mu, t)}{(\mu-1)!} \right) - 2^{-\mu} e^{2t} \left(1 - \frac{\gamma(\mu, 2t)}{(\mu-1)!} \right).$$

Multiplying by e^{-t} and simplifying gives

$$F_{L_{\mu}^{c*}}(x) = 1 + 2^{\mu} e^{-t} P(\mu, t) - P(\mu, 2t)$$

where $P(\mu, z) = \gamma(\mu, z)/(\mu-1)!$. Re-substituting $t = (x/s)^{-\alpha}$ yields equation (3). The PDF $f_{L_{\mu}^{c*}}(x)$ follows

directly by substituting into the corresponding density expression.

For the upper current records, substituting again by (1) and (2) in (9) and (10) and with similar steps as before, it gives (5) and (6).

Houchens (1984) also found the joining PDF between two lower current and two upper current records as

$$f_{L_{\mu}^{C*}, L_{\nu}^{C*}}(l_{\mu}, l_{\nu}) = \frac{f(l_{\mu})f(l_{\nu})}{F(l_{\mu})^{(v-\mu-1)!}} \left(H(l_{\mu}) - H(l_{\nu}) \right)^{v-\mu-1}, l_{\nu} < l_{\mu}, 1 < \mu < \nu \quad (11)$$

Where $H(x) = \log F(x)$. And

$$f_{U_{\mu}^{C*}, U_{\nu}^{C*}}(u_{\mu}, u_{\nu}) = \frac{f(u_{\mu})f(u_{\nu})}{(1-F(u_{\mu}))^{(v-\mu-1)!}} \left(R(u_{\mu}) - R(u_{\nu}) \right)^{v-\mu-1}, u_{\nu} > u_{\mu}, 1 < \mu < \nu. \quad (12)$$

Where $R(x) = -\log(1 - F(x))$.

From (11) and (12), we can conclude the next conditional probability of $L_{\nu}^{C*} | L_{\mu}^{C*} = l_{\mu}$ and $U_{\nu}^{C*} | U_{\mu}^{C*} = u_{\mu}$ as follows

$$f_{L_{\mu}^{C*} | L_{\mu}^{C*}}(l_{\nu} | l_{\mu}) = \frac{f(l_{\nu})}{F(l_{\mu})^{(v-\mu-1)!}} \left(H(l_{\mu}) - H(l_{\nu}) \right)^{v-\mu-1}, \quad (13)$$

and

$$f_{U_{\nu}^{C*} | U_{\mu}^{C*}}(u_{\nu} | u_{\mu}) = \frac{f(u_{\nu})}{(1-F(u_{\mu}))^{(v-\mu-1)!}} \left(R(u_{\nu}) - R(u_{\mu}) \right)^{v-\mu-1}. \quad (14)$$

To demonstrate the shapes of the upper and lower current records' PDFs and CDFs found in Theorem 1, different values of α, s , and μ were tested, and the results are presented in Figures 1-4.

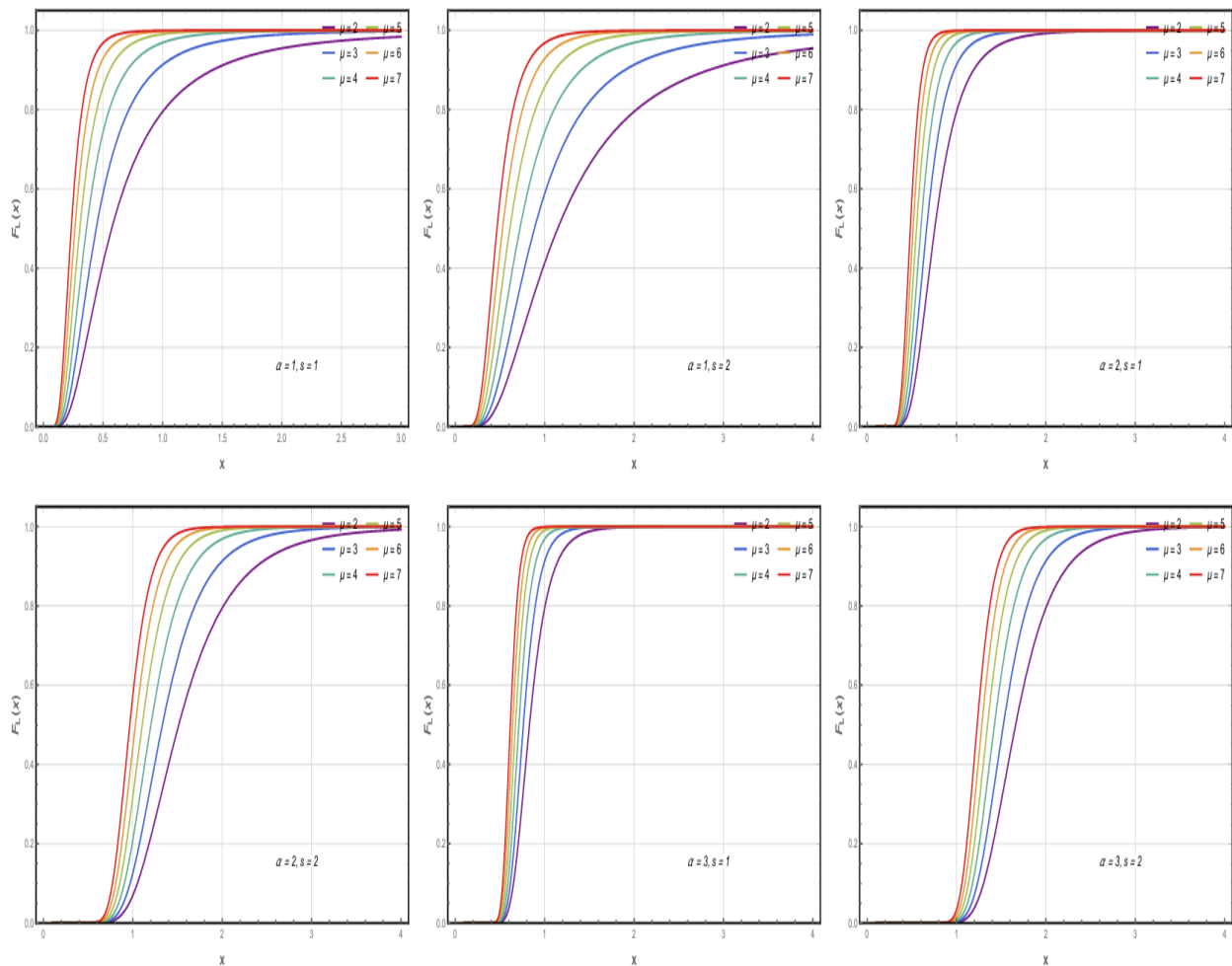


Figure 1: Plot of $F_{L_{\mu}^{C*}}(x; \alpha, s)$ for Different Values of μ, α , and s

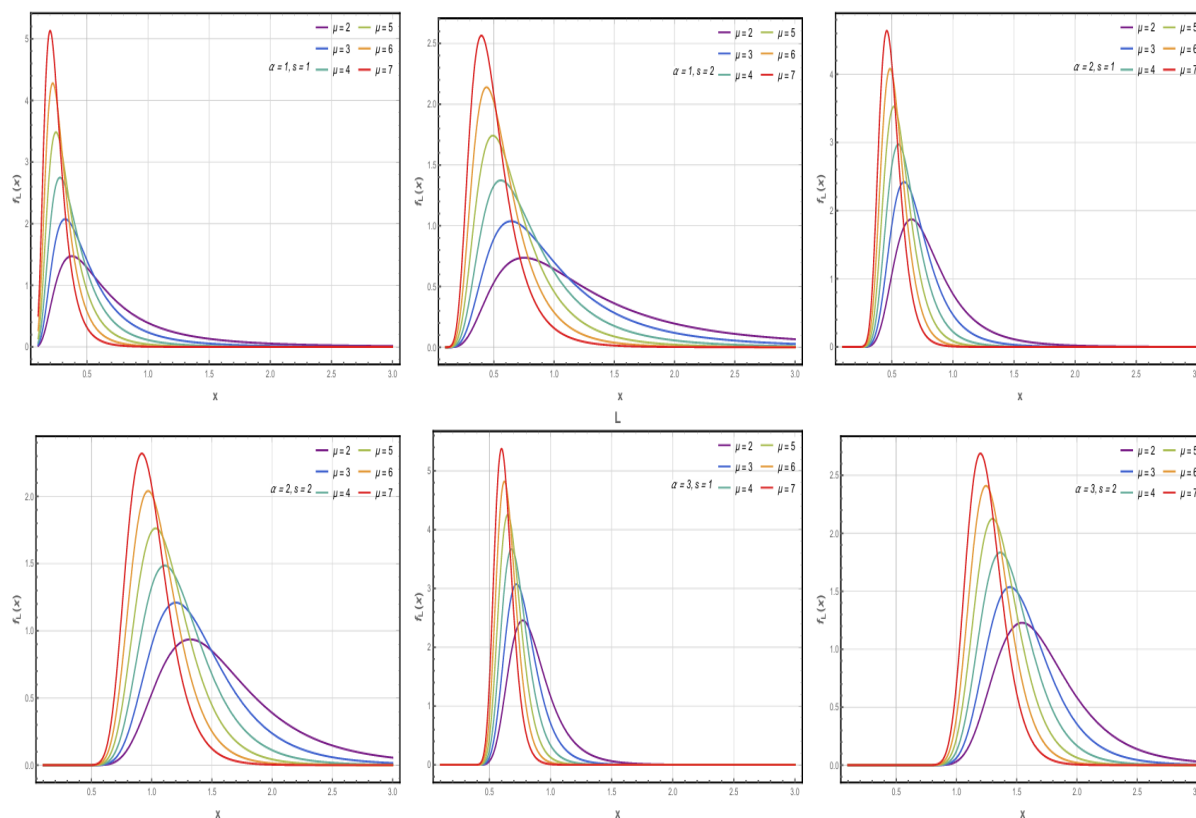


Figure 2: Plot of $f_{L\mu}^*(x; \alpha, s)$ for Different Values of μ, α , and s

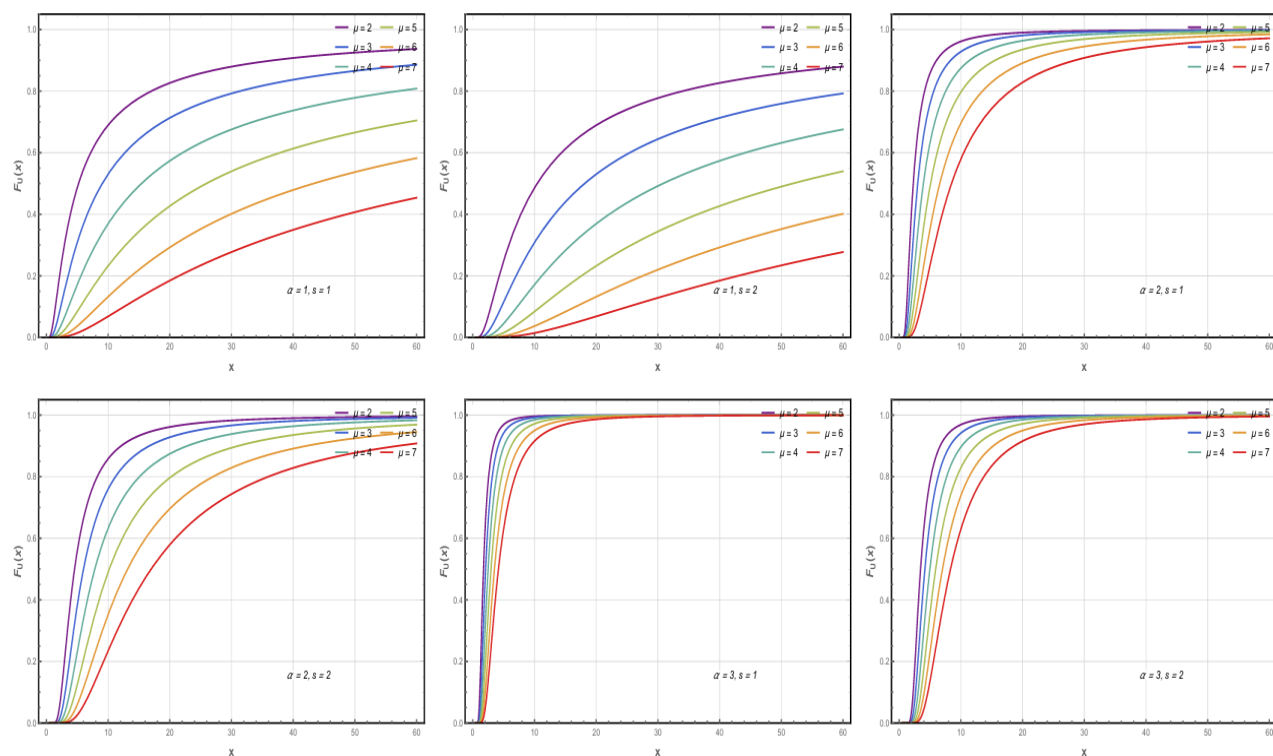


Figure 3: Plot of $F_{U\mu}^*(x; \alpha, s)$ for Different Values of μ, α , and s

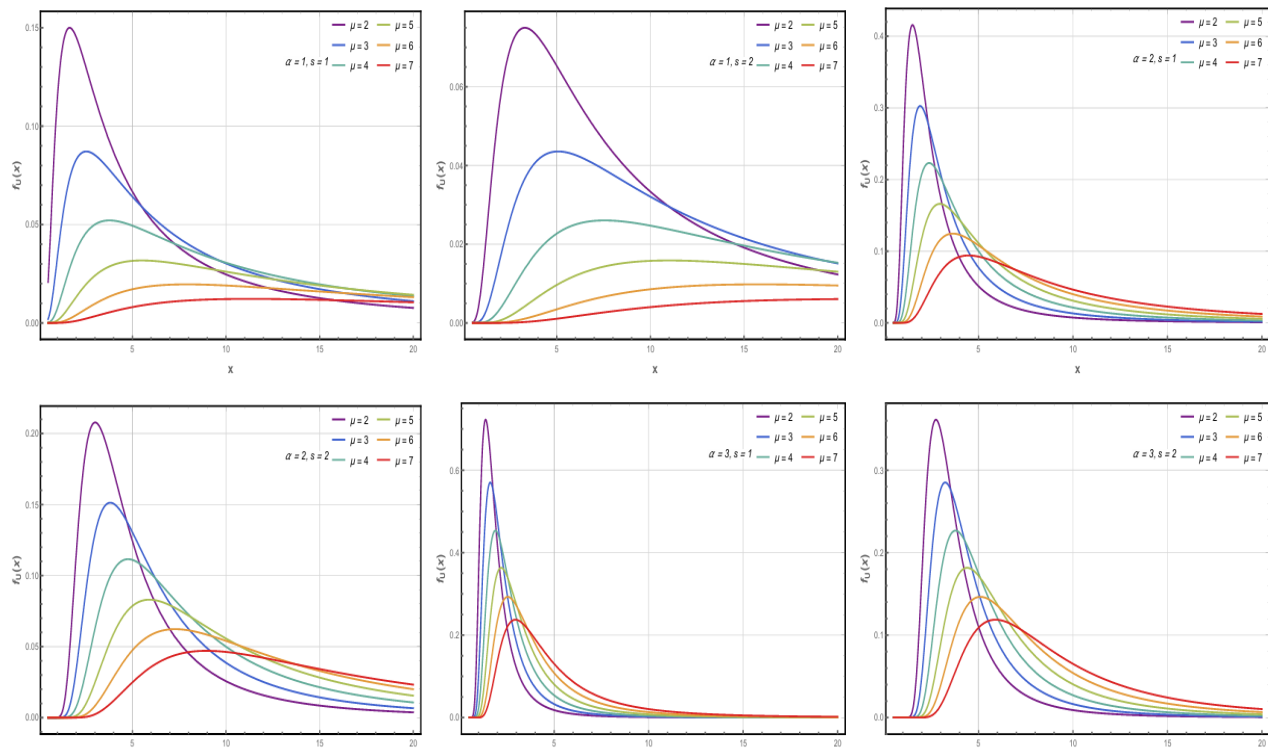


Figure 4: Plot of $f_{U_{\mu}^s}(x; \alpha, s)$ for Different Values of μ, α , and s

3. Moments of Fréchet Current Records

Moments play a fundamental role in statistics as they summarise key characteristics of a probability distribution. They offer valuable insights into the central tendency, dispersion, skewness, and kurtosis of the data, thereby helping to understand the distribution's overall behavior and shape. In this section, we derive explicit expressions for the r -th moments Δ^r of both the lower and upper current record values when the underlying data follow a Fréchet distribution. These moment expressions are essential for applications such as reliability analysis, risk assessment, and probabilistic modeling, as they quantify the expected magnitude and variability of current record-breaking observations.

Theorem 2. When $\mu > 1$, the r -th moment of the lower current Fréchet record $\Delta_{L_{\mu}^{C^*}}^r$ and for the upper current record $\Delta_{U_{\mu}^{C^*}}^r$ can be calculated receptively using the following closed forms

$$\Delta_{L_{\mu}^{C^*}}^r = 2^{\mu} s^r \left[\Gamma\left(1 - \frac{r}{\alpha}\right) - 2^{\frac{r}{\alpha}-1} \sum_{k=0}^{\mu-1} \frac{\Gamma\left(k + 1 - \frac{r}{\alpha}\right)}{2^k k!} \right]. \quad (15)$$

$$\Delta_{U_{\mu}^{C^*}}^r = 2^{\mu} s^r \Gamma\left(1 - \frac{r}{\alpha}\right) \left[1 - \sum_{k=0}^{\mu-1} \frac{1}{k!} \sum_{p=0}^{\infty} a_p(k) \left((k+p+1)^{\frac{r}{\alpha}-1} - (k+p+2)^{\frac{r}{\alpha}-1} \right) \right] \quad (16)$$

Where $a_p(k)$ are the coefficients from the expansion:

$$[-\log(1-t)]^k = \sum_{p=0}^{\infty} a_p(k) t^{k+p}, |t| < 1. \quad (17)$$

Proof. $\Delta_{L_{\mu}^{C^*}}^r$ is defined as

$$\Delta_{L_{\mu}^{C^*}}^r = \int_0^{\infty} x^r f_{L_{\mu}^{C^*}}(x; \alpha, s) dx.$$

Upon using (8) and changing of variable

$$t = (x/s)^{-\alpha} \Rightarrow x = st^{-1/\alpha}, dx = -\frac{s}{\alpha} t^{-1/\alpha-1} dt,$$

then

$$x^r \left(\frac{x}{s}\right)^{-(\alpha+1)} dx = s^r t^{-r/\alpha} \left(-\frac{s}{\alpha} dt\right) \frac{\alpha}{s} = s^r t^{-r/\alpha} dt,$$

we get

$$\Delta_{L_{\mu}^{c*}}^r = 2^\mu s^r \int_0^\infty t^{-r/\alpha} e^{-t} P(\mu, t) dt.$$

Now, expanding the regularized incomplete gamma function for integer μ ,

$$P(\mu, t) = \frac{\gamma(\mu, t)}{(\mu-1)!} = 1 - \frac{\Gamma(\mu, t)}{(\mu-1)!}.$$

and substitute into the integral, we get:

$$\Delta_{L_{\mu}^{c*}}^r = 2^\mu s^r \int_0^\infty t^{-r/\alpha} e^{-t} \left[1 - \frac{\Gamma(\mu, t)}{(\mu-1)!}\right] dt = 2^\mu s^r \left[\underbrace{\int_0^\infty t^{-r/\alpha} e^{-t} dt}_{I_1} - \frac{1}{(\mu-1)!} \underbrace{\int_0^\infty t^{-r/\alpha} e^{-t} \Gamma(\mu, t) dt}_{I_2} \right].$$

I_1 is simply the standard gamma integral:

$$I_1 = \int_0^\infty t^{-r/\alpha} e^{-t} dt = \Gamma\left(1 - \frac{r}{\alpha}\right),$$

while for I_2 we have to expand $\Gamma(\mu, t)$ for integer μ . We have

$$\Gamma(\mu, t) = (\mu-1)! e^{-t} \sum_{k=0}^{\mu-1} \frac{t^k}{k!}$$

Thus,

$$I_2 = \int_0^\infty t^{-r/\alpha} e^{-t} (\mu-1)! e^{-t} \sum_{k=0}^{\mu-1} \frac{t^k}{k!} dt = (\mu-1)! \sum_{k=0}^{\mu-1} \frac{1}{k!} \int_0^\infty t^{k-r/\alpha} e^{-2t} dt$$

Evaluate the integral inside the sum.

$$\int_0^\infty t^{k-r/\alpha} e^{-2t} dt = 2^{-k-1+r/\alpha} \Gamma(k+1-r/\alpha)$$

and combine with $1/(\mu-1)!$, the factor $(\mu-1)!$ cancels, giving

$$\frac{I_2}{(\mu-1)!} = \sum_{k=0}^{\mu-1} \frac{\Gamma(k+1-r/\alpha)}{2^{k+1-r/\alpha} k!} = 2^{r/\alpha-1} \sum_{k=0}^{\mu-1} \frac{\Gamma(k+1-r/\alpha)}{2^k k!}.$$

Substituting with the final form of I_1 and I_2 , we get (15). For $\Delta_{U_{\mu}^{c*}}^r$, it is defined as

$$\Delta_{U_{\mu}^{c*}}^r = \int_0^\infty x^r f_{U_{\mu}^{c*}}(x; \alpha, s) dx,$$

Using (6) and change the variables as $t = (x/s)^{-\alpha}$ again, we reach

$$\Delta_{U_{\mu}^{c*}}^r = 2^\mu s^r \int_0^\infty t^{-r/\alpha} e^{-t} P(\mu, -\log(1 - e^{-t})) dt.$$

Expanding $P(\mu, z)$, we get

$$\Delta_{U_{\mu}^{C^*}}^r = 2^{\mu} s^r \int_0^{\infty} t^{-r/\alpha} e^{-t} dt - 2^{\mu} s^r \sum_{k=0}^{\mu-1} \frac{1}{k!} \int_0^{\infty} t^{-r/\alpha} e^{-t} (1 - e^{-t}) [-\log(1 - e^{-t})]^k dt.$$

We will now use the Series expansion of $[-\log(1 - e^{-t})]^k$ to find a closed form for the second integral. Using the coefficients $a_p(k)$ from (17) we reach

$$\int_0^{\infty} t^{-r/\alpha} e^{-t} (1 - e^{-t}) [-\log(1 - e^{-t})]^k dt = \Gamma\left(1 - \frac{r}{\alpha}\right) \sum_{p=0}^{\infty} a_p(k) [(k + p + 1)^{r/\alpha-1} - (k + p + 2)^{r/\alpha-1}]$$

Combine all terms, (17) will be concluded.

4. Estimation of Parameters

In most practical situations, the underlying distribution is characterized by unknown parameters, most notably the shape parameter α and the scale parameter s . Accurate estimation of these parameters is essential for making reliable statistical inferences, conducting risk assessment, and performing predictions. Since the true values of α and s are rarely available in real-world applications, they must be inferred from the observed data. Among the various inferential techniques, maximum likelihood estimation and Bayesian estimation are two widely used and powerful approaches. The maximum likelihood estimation method provides point estimates by maximizing the likelihood function constructed from the observed data. In contrast, the Bayesian approach incorporates prior information about the parameters and updates it based on the observed sample. Both estimation frameworks can be suitably modified to accommodate current record data, allowing efficient parameter estimation even when only record observations are available.

4.1 Non-Bayesian Estimation (NBE)

One of the most widely used NBE techniques for parameter estimation is maximum likelihood estimation (MLE), which will be applied here. Let $L_1^{C^*}, L_2^{C^*}, \dots, L_{\mu}^{C^*}$ denote the μ th lower current record. The joint PDF of them is defined by Barakat et al. (2014) to be

$$f_{L_1^{C^*}, L_2^{C^*}, \dots, L_{\mu}^{C^*}}(x_1, x_2, \dots, x_{\mu}) = \frac{\bar{F}^2(x_{\mu}; \alpha, s) \bar{F}(x_1; \alpha, s)}{F(x_1; \alpha, s)} \left(\prod_{j=1}^{\mu} \frac{2f(x_j; \alpha, s)}{\bar{F}(x_j; \alpha, s)} \right), x_1 > x_2 > \dots > x_{\mu}. \quad (18)$$

And if $U_1^{C^*}, U_2^{C^*}, \dots, U_{\mu}^{C^*}$ denote the first μ th upper current record values from this sequence, then the joint PDF of them will be

$$f_{U_1^{C^*}, U_2^{C^*}, \dots, U_{\mu}^{C^*}}(x_1, x_2, \dots, x_{\mu}) = \frac{\bar{F}^2(x_{\mu}; \alpha, s) F(x_1; \alpha, s)}{F(x_1; \alpha, s)} \left(\prod_{j=1}^{\mu} \frac{2f(x_j; \alpha, s)}{\bar{F}(x_j; \alpha, s)} \right), x_1 < x_2 < \dots < x_{\mu}. \quad (19)$$

The likelihood function is proportional to the associated joint distribution, as it depends solely on the observed sequence of current records. The likelihood function of a set of lower current records $\underline{L}^{C^*} = (L_1^{C^*}, L_2^{C^*}, \dots, L_{\mu}^{C^*})$ based on (18) will be

$$L(\alpha, s | \underline{L}^{C^*}) = (2\alpha)^{\mu} \left[\exp\left(\left(\frac{x_1}{s}\right)^{-\alpha}\right) - 1 \right] \left[1 - \exp\left(\left(\frac{x_{\mu}}{s}\right)^{-\alpha}\right) \right]^2 s^{\mu\alpha} \prod_{j=1}^{\mu} (x_j)^{-(\alpha+1)}. \quad (20)$$

Then, the log-likelihood function will be

$$\begin{aligned} \ell(\alpha, s | \underline{L}^{C^*}) &= \mu \log(2\alpha) + \mu \alpha \log s - (\alpha + 1) \sum_{j=1}^{\mu} \log x_j + \log \left[\exp\left(\left(\frac{x_1}{s}\right)^{-\alpha}\right) - 1 \right] \\ &\quad + 2 \log \left[1 - \exp\left(-\left(\frac{x_{\mu}}{s}\right)^{-\alpha}\right) \right]. \end{aligned} \quad (21)$$

Differentiation (21) with respect to α and then s , and equal to zero, we get

$$\frac{\mu}{\alpha} + \mu \log s - \sum_{j=1}^{\mu} \log x_j - \frac{\left(\frac{x_1}{s}\right)^{-\alpha} \log\left(\frac{x_1}{s}\right) \exp\left[\left(\frac{x_1}{s}\right)^{-\alpha}\right]}{\exp\left[\left(\frac{x_1}{s}\right)^{-\alpha}\right] - 1} + \frac{2 \left(\frac{x_{\mu}}{s}\right)^{-\alpha} \log\left(\frac{x_{\mu}}{s}\right) \exp\left[-\left(\frac{x_{\mu}}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{x_{\mu}}{s}\right)^{-\alpha}\right]} = 0, \quad (22)$$

$$\frac{\mu\alpha}{s} + \frac{\alpha \left(\frac{x_1}{s}\right)^{-\alpha} \exp\left[\left(\frac{x_1}{s}\right)^{-\alpha}\right]}{\exp\left[\left(\frac{x_1}{s}\right)^{-\alpha}\right] - 1} - \frac{2\alpha \left(\frac{x_\mu}{s}\right)^{-\alpha} \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]} = 0. \quad (23)$$

Now, to obtain the likelihood function of a set of upper current records $\underline{U}^{C*} = (U_1^{C*}, U_2^{C*}, \dots, U_\mu^{C*})$ based on (19), it will be

$$L(\alpha, s \mid \underline{U}^{C*}) = \left(1 - \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]\right)^2 \cdot \frac{\exp\left[-\left(\frac{x_1}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{x_1}{s}\right)^{-\alpha}\right]} (2\alpha)^\mu s^{\mu\alpha} \prod_{j=1}^\mu \frac{(x_j)^{-(\alpha+1)} \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]} \quad (24)$$

With a log-likelihood function

$$\begin{aligned} \ell(\alpha, s \mid \underline{U}^{C*}) = & 2\log\left(1 - \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]\right) - \log\left(1 - \exp\left[-\left(\frac{x_1}{s}\right)^{-\alpha}\right]\right) - \left(\frac{x_1}{s}\right)^{-\alpha} + \mu\log(2\alpha) + \mu\alpha\log s \\ & - (\alpha + 1) \sum_{j=1}^\mu \log(x_j) - \sum_{j=1}^\mu \left(\frac{x_j}{s}\right)^{-\alpha} - \sum_{j=1}^\mu \log\left(1 - \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]\right), \end{aligned} \quad (25)$$

and when differentiating (25) with respect to α and then s and setting equal to zero, we reach

$$\begin{aligned} & \frac{2\exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right] - 1}{1 - \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]} \cdot (\log s - \log x_\mu) \cdot \left(\frac{x_\mu}{s}\right)^{-\alpha} - \frac{2(\log s - \log x_1)}{1 - \exp\left[-\left(\frac{x_1}{s}\right)^{-\alpha}\right]} \cdot \left(\frac{x_1}{s}\right)^{-\alpha} \\ & - \sum_{j=2}^{\mu-1} \frac{\log s - \log x_j}{1 - \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]} \cdot \left(\frac{x_j}{s}\right)^{-\alpha} + \frac{\mu}{\alpha} + \mu\log s - \sum_{j=1}^\mu \log x_j = 0, \end{aligned} \quad (26)$$

and

$$\sum_{j=2}^{\mu-1} \frac{\alpha \left(\frac{x_j}{s}\right)^{-\alpha} \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]}{s \left(1 - \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]\right)} + 3 \frac{\alpha \left(\frac{x_\mu}{s}\right)^{-\alpha} \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]}{s \left(1 - \exp\left[-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right]\right)} + \frac{\alpha}{s} \sum_{j=2}^\mu \left(\frac{x_j}{s}\right)^{-\alpha} + \frac{\alpha\mu}{s} = 0. \quad (27)$$

The MLEs $\hat{\alpha}_{L^{C*}}^{ML}$ from (22), $\hat{s}_{L^{C*}}^{ML}$ from (23), $\hat{\alpha}_{U^{C*}}^{ML}$ from (26), and $\hat{s}_{U^{C*}}^{ML}$ from (27), are obtained by solving the corresponding nonlinear equations derived from the likelihood function. Due to the complexity of these equations, closed-form solutions are generally not available. Consequently, numerical optimisation techniques, such as the Newton–Raphson method or the Fisher scoring algorithm, are used to iteratively approximate parameter values. These methods leverage successive linear approximations or expected information matrices to efficiently converge to the MLEs, providing reliable estimates for practical applications.

4.2 Bayesian Estimation (BE)

BE plays a crucial role in statistical inference by systematically combining prior information with observed data to obtain more informative and adaptable parameter estimates. In contrast to classical approaches that rely solely on sample data, the Bayesian framework updates prior beliefs via the posterior distribution, thereby enabling direct, explicit quantification of uncertainty. This characteristic makes Bayesian methods particularly effective in settings with small sample sizes, complex model structures, or limited expert knowledge. Consequently, BE offers a coherent and robust framework for inference, prediction, and decision-making in a broad range of theoretical and applied disciplines.

To obtain the BEs for the parameters of the lower-current record distribution, suitable prior distributions are assigned to the unknown parameters. Specifically, independent gamma-type priors are assumed for the shape and scale parameters, respectively, as

$$\pi(\alpha) \propto \alpha^{a_1-1} e^{-b_1\alpha}, \quad \pi(s) \propto s^{a_2-1} e^{-b_2s}.$$

By combining these prior distributions with the likelihood function given in (20), the joint posterior distribution of α and s , conditional on the observed lower current record sample $\underline{L}^{C*}$, is obtained up to a normalizing constant as

$$\pi(\alpha, s \mid \underline{L}^{C*}) \propto \alpha^{\mu+a_1-1} s^{\mu\alpha+a_2-1} \left[\exp\left(\left(\frac{x_1}{s}\right)^{-\alpha}\right) - 1 \right] \left[1 - \exp\left(-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right) \right]^2$$

$$\times e^{-b_1\alpha - b_2s} \prod_{j=1}^{\mu} (x_j)^{-(\alpha+1)}. \quad (28)$$

Under the squared error loss (SEL) function, the BEs of the parameters α and s , denoted by $\hat{\alpha}_{L^{c*}}^{BE}$ and $\hat{s}_{L^{c*}}^{BE}$, respectively, are given by the posterior expectations. For the observed lower current record sequence $\underline{L}^{c*}$, These estimators can be expressed as

$$E(\alpha | \underline{L}^{c*}) = \int_0^\infty \int_0^\infty \alpha \pi(\alpha, s | \underline{L}^{c*}) d\alpha ds, \quad (29)$$

$$E(s | \underline{L}^{c*}) = \int_0^\infty \int_0^\infty s \pi(\alpha, s | \underline{L}^{c*}) d\alpha ds \quad (30)$$

Now, to derive the BEs of the parameters based on the upper current record distribution, independent prior distributions are assigned to the unknown parameters. In particular, gamma-type priors are assumed for the shape and scale parameters, respectively, given by

$$\pi(\alpha) \propto \alpha^{c_1-1} e^{-d_1\alpha}, \quad \pi(s) \propto s^{c_2-1} e^{-d_2s}.$$

By combining these prior distributions with the likelihood function of the upper current records given in (24), the joint posterior distribution of α and s , conditional on the observed upper current record sample $\underline{U}^{c*}$, is obtained up to a proportionality constant as

$$\begin{aligned} \pi(\alpha, s | \underline{U}^{c*}) &\propto \alpha^{\mu+c_1-1} s^{\mu\alpha+c_2-1} \left(1 - \exp\left(-\left(\frac{x_\mu}{s}\right)^{-\alpha}\right)\right)^2 \frac{\exp\left[-\left(\frac{x_1}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{x_1}{s}\right)^{-\alpha}\right]} \\ &\times \prod_{j=1}^{\mu} \frac{(x_j)^{-(\alpha+1)} \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{x_j}{s}\right)^{-\alpha}\right]} e^{-d_1\alpha - d_2s}. \end{aligned} \quad (31)$$

Under the SEL framework, the BEs of the parameters α and s , represented by $\hat{\alpha}_{U^{c*}}^{BE}$ and $\hat{s}_{U^{c*}}^{BE}$, respectively, are obtained by taking the means of their posterior distributions. For the observed upper current record sample $\underline{U}^{c*}$, these estimators are given by

$$E(\alpha | \underline{U}^{c*}) = \int_0^\infty \int_0^\infty \alpha \pi(\alpha, s | \underline{U}^{c*}) d\alpha ds, \quad (32)$$

$$E(s | \underline{U}^{c*}) = \int_0^\infty \int_0^\infty s \pi(\alpha, s | \underline{U}^{c*}) d\alpha ds.$$

Obtaining closed-form expressions for (29), (30), (32), and (33) is analytically intractable due to the highly complex structures of the posterior distributions in (20) and (24). Consequently, numerical techniques must be employed to evaluate these integrals. In particular, simulation-based approaches such as Markov Chain Monte Carlo (MCMC) methods are adopted, with the Metropolis-Hastings (MH) algorithm being utilized to obtain approximate Bayesian estimates.

5. Current Records Future Prediction

By employing prediction intervals specifically designed for the type of current record under consideration, and by utilizing carefully refined estimators that account for the underlying data structure, researchers can produce forecasts that are both more accurate and more informative. This approach not only enhances the reliability of statistical inference but also strengthens the overall robustness and interpretability of predictive models, thereby facilitating better decision-making and more effective risk assessment across diverse applied and theoretical contexts.

5.1 Non-Bayesian Prediction Intervals (NBPI)

The prediction of future lower or upper current records, as well as future record ranges, was first systematically addressed by Barakat et al. (2014), who developed prediction intervals for these quantities. These intervals are constructed based on the most recently observed current record and a calculated term that depends on the type of current record being considered—whether lower, upper, or a record range. Within these intervals, a quantity denoted by $\Omega_{v,\gamma}$ appears, where v indicates the position of the future current record to be predicted. Specifically, $v = 1$ corresponds to the first predicted value, $v = 2$ to the second, and so forth, up to a maximum of five predicted values, while γ represents the chosen significance level. Values of $\Omega_{v,\gamma}$ have been tabulated in the literature for reference.

In our methodology, we modify the interval construction by estimating the parameters α and s based on the type of interval. For intervals concerning upper current records, we use $\hat{\alpha}_{U^{c*}}^{ML}$ or $\hat{\alpha}_{U^{c*}}^{BE}$ in place of the original estimator $\hat{\alpha}$ computed from the full dataset, and $\hat{s}_{U^{c*}}^{ML}$ or $\hat{s}_{U^{c*}}^{BE}$ instead of the original $\hat{s}$. Conversely, for intervals involving lower current records, we employ $\hat{\alpha}_{L^{c*}}^{ML}$ or $\hat{\alpha}_{L^{c*}}^{BE}$ and $\hat{s}_{L^{c*}}^{ML}$ or $\hat{s}_{L^{c*}}^{BE}$. This adjustment improves the accuracy and reliability of the resulting predictive intervals.

The following algorithm summarizes the steps for the prediction procedure:

Step (1) For the given dataset, identify the probability distribution F that best describes the observed data.

Step (2) Extract the observed lower and upper current record values and compute the associated record ranges. Let μ denote the number of observed lower and upper current records.

Step (3) Estimate $\hat{\alpha}_{L^{c*}}^{ML}$ and $\hat{s}_{L^{c*}}^{ML}$ or $\hat{\alpha}_{L^{c*}}^{BE}$ and $\hat{s}_{L^{c*}}^{BE}$ using the approaches outlined in Subsections 4.1 and 4.2. Similarly, estimate $\hat{\alpha}_{U^{c*}}^{ML}$ and $\hat{s}_{U^{c*}}^{ML}$ or $\hat{\alpha}_{U^{c*}}^{BE}$ and $\hat{s}_{U^{c*}}^{BE}$.

Step (4) Determine the position ν of the future current record to be predicted and the desired significance level γ . Then obtain the corresponding value of $\Omega_{\nu,\gamma}$ from the tables provided by Barakat et al. (2014).

Step (5) Using the estimates from Step (3), compute the lower and upper bounds of the predictive confidence intervals as follows:

1. $\left(U_{\mu}^{c*}, F^{-1}\left(1 - \bar{F}^{1+\Omega_{\nu,\gamma}}(U_{\mu}^{c*})\right) \right)$ is a $(1 - \gamma)\%$ confidence interval for $U_{\mu+\nu}^{c*}$.
2. $\left(F^{-1}\left(F^{1+\Omega_{\nu,\gamma}}(L_{\mu}^{c*})\right), L_{\mu}^{c*} \right)$ is a $(1 - \gamma)\%$ confidence interval for $L_{\mu+\nu}^{c*}$.
3. $\left(U_{\mu}^{c*} - L_{\mu}^{c*}, F^{-1}\left(1 - \bar{F}^{1+\Omega_{\nu,\gamma}}(U_{\mu}^{c*})\right) - F^{-1}\left(F^{1+\Omega_{\nu,\gamma}}(L_{\mu}^{c*})\right) \right)$ is a $\beta\%$ confidence interval for $R_{\mu+\nu}^{c*}$, where $\beta \geq \max(1 - 2\gamma, 0)$.

5.2 Bayesian Predictive Interval (BPI) and Bayesian Predictive Value (BPV)

Finding BPV using Bayesian predictive methodology (BPM) is a statistical framework based on Bayes' theorem, which combines prior knowledge about unknown parameters with information extracted from observed data to make informed predictions. Unlike classical approaches that produce single-point estimates, BPM incorporates uncertainty by integrating over the full posterior distribution of the parameters. This integration yields a Bayesian predictive distribution (BPD) for future observations, which naturally accounts for both the inherent variability in the data and the uncertainty in parameter estimates. From this predictive distribution, BPIs can be constructed, providing a range of plausible values for future observations along with a specified probability of coverage. As additional data become available, both the predictive distributions and the corresponding BPIs are consistently and coherently updated, making BPM particularly advantageous in situations with limited data or when prior knowledge plays a crucial role.

For lower current records, to find BPV of future observations proceeds, let L_{ν}^{c*} denote the ν th lower current record to be predicted, with $\nu > \mu$, where μ is the number of observed lower current records. Conditional on the observed sequence $\underline{L}^{c*}$, the posterior predictive density of L_{ν}^{c*} is defined as

$$f_{L_{\nu}^{c*}|\underline{L}^{c*}}(l_{\nu}) = \int_0^{\infty} \int_0^{\infty} f_{L_{\nu}^{c*}|L_{\mu}^{c*}}(l_{\nu} | l_{\mu}) \pi(\alpha, s | \underline{L}^{c*}) ds d\alpha. \quad (34)$$

The conditional density $f_{L_{\nu}^{c*}|L_{\mu}^{c*}}(l_{\nu} | l_{\mu})$ follows from Eq. (13). Using the Fréchet PDF and CDF given in Eqs. (1) – (2), it can be expressed as

$$f_{L_{\nu}^{c*}|L_{\mu}^{c*}}(l_{\nu} | l_{\mu}) = \frac{\alpha}{s(\nu - \mu - 1)!} \left(\frac{l_{\nu}}{s}\right)^{-(\alpha+1)} \exp \left[-\left(\frac{l_{\nu}}{s}\right)^{-\alpha} \right] \exp \left[\left(\frac{l_{\mu}}{s}\right)^{-\alpha} \right] \\ \times \left[\left(\frac{l_{\nu}}{s}\right)^{-\alpha} - \left(\frac{l_{\mu}}{s}\right)^{-\alpha} \right]^{\nu-\mu-1}, 0 < l_{\nu} < l_{\mu}. \quad (35)$$

Substituting (35) together with the joint posterior distribution in (29) into (34) yields the posterior predictive density

$$f_{L_{\nu}^{c*}|\underline{L}^{c*}}(l_{\nu}) = \int_0^{\infty} \int_0^{\infty} \alpha^{\mu+\alpha_1} s^{-\mu+\alpha_2-2} \left(\frac{l_{\nu}}{s}\right)^{-(\alpha+1)} \left[\left(\frac{l_{\nu}}{s}\right)^{-\alpha} - \left(\frac{l_{\mu}}{s}\right)^{-\alpha} \right]^{\nu-\mu-1} \\ \times \left[\exp \left(\left(\frac{x_1}{s}\right)^{-\alpha} \right) - 1 \right] \left[1 - \exp \left(-\left(\frac{x_{\mu}}{s}\right)^{-\alpha} \right) \right]^2 \prod_{j=1}^{\mu} \left(\frac{x_j}{s}\right)^{-(\alpha+1)}$$

$$\times \exp \left\{ -\left(\frac{l_v}{s}\right)^{-\alpha} + \left(\frac{l_\mu}{s}\right)^{-\alpha} - b_1\alpha - b_2s \right\} ds d\alpha. \quad (36)$$

Under SEL, the PBV of L_v^{c*} is the posterior predictive mean

$$\widehat{L_v^{c*}}^{BPV} = E(L_v^{c*} | \underline{L^{c*}}) = \int_0^{l_\mu} l_v f_{L_v^{c*} | \underline{L^{c*}}}^p(l_v) dl_v. \quad (37)$$

Since a closed-form evaluation of (37) is not analytically tractable, a Monte Carlo approximation is employed:

$$\widehat{L_v^{c*}}^{BPV} \approx \frac{1}{K} \sum_{k=1}^K L_v^{c*(k)},$$

where $\{L_v^{c*(k)}\}_{k=1}^K$ are generated from the posterior predictive distribution via MCMC sampling. A $100(1 - \gamma)\%$ Bayesian predictive interval (L_γ, U_γ) for L_v^{c*} satisfies

$$P(L_\gamma \leq L_v^{c*} \leq U_\gamma | L^{c*}) = 1 - \gamma.$$

The predictive limits are determined by

$$\int_0^{L_\gamma} f_{L_v^{c*} | \underline{L^{c*}}}^p(l_v) dl_v = \gamma/2 \quad (38)$$

$$\int_0^{U_\gamma} f_{L_v^{c*} | \underline{L^{c*}}}^p(l_v) dl_v = 1 - \gamma/2. \quad (39)$$

In practice, these bounds are approximated using the ordered posterior predictive sample $\{L_v^{c*(k)}\}_{k=1}^K$ as

$$(L_\gamma, U_\gamma) = (q_{\gamma/2}, q_{1-\gamma/2}),$$

where q_p denotes the p th quantile of the posterior predictive distribution.

Since lower current records form a decreasing sequence, $L_v^{c*} < L_\mu^{c*}$ for $v > \mu$, all BPI are necessarily upper-bounded by the most recent observed lower record l_μ . This constraint is enforced in both simulation and interpretation.

Now let U_v^{c*} denote the v th future upper current record to be predicted, where $v > \mu$ and μ is the number of observed upper current records. Conditional on the observed sequence $\underline{U^{c*}}$, the posterior predictive density of U_v^{c*} is defined by

$$f_{U_v^{c*} | \underline{U^{c*}}}^p(\mathcal{U}_v) = \int_0^\infty \int_0^\infty f_{U_v^{c*} | U_\mu^{c*}}(\mathcal{U}_v | \mathcal{U}_\mu) \pi(\alpha, s | \underline{U^{c*}}) ds d\alpha, \quad (40)$$

where the conditional density $f_{U_v^{c*} | U_\mu^{c*}}(\mathcal{U}_v | \mathcal{U}_\mu)$ is given in (14). Using (1) (2), the conditional density in (14) can be written explicitly as

$$\begin{aligned} f_{U_v^{c*} | U_\mu^{c*}}(\mathcal{U}_v | \mathcal{U}_\mu) &= \frac{\alpha}{s(v-\mu-1)!} \left(\frac{\mathcal{U}_v}{s}\right)^{-(\alpha+1)} \exp\left[-\left(\frac{\mathcal{U}_v}{s}\right)^{-\alpha}\right] \left[1 - \exp\left(-\left(\frac{\mathcal{U}_v}{s}\right)^{-\alpha}\right)\right]^1 \\ &\quad \times \left[\left(\frac{\mathcal{U}_\mu}{s}\right)^{-\alpha} - \left(\frac{\mathcal{U}_v}{s}\right)^{-\alpha}\right]^{v-\mu-1}, \mathcal{U}_v > \mathcal{U}_\mu \end{aligned} \quad (41)$$

Substituting (41) and the joint posterior density of (α, s) given in (32) into (40) gives the posterior predictive density of the future upper current record:

$$\begin{aligned} f_{U_v^{c*} | \underline{U^{c*}}}^p(\mathcal{U}_v) &= \int_0^\infty \int_0^\infty \alpha^{\mu+c_1} s^{-\mu+c_2-2} \left(\frac{\mathcal{U}_v}{s}\right)^{-(\alpha+1)} \left[\left(\frac{\mathcal{U}_\mu}{s}\right)^{-\alpha} - \left(\frac{\mathcal{U}_v}{s}\right)^{-\alpha}\right]^{v-\mu-1} \\ &\quad \times \left(1 - \exp\left[-\left(\frac{\mathcal{U}_\mu}{s}\right)^{-\alpha}\right]\right)^2 \frac{\exp\left[-\left(\frac{\mathcal{U}_1}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{\mathcal{U}_1}{s}\right)^{-\alpha}\right]} \prod_{j=1}^\mu \frac{\left(\frac{\mathcal{U}_j}{s}\right)^{-(\alpha+1)} \exp\left[-\left(\frac{\mathcal{U}_j}{s}\right)^{-\alpha}\right]}{1 - \exp\left[-\left(\frac{\mathcal{U}_j}{s}\right)^{-\alpha}\right]} \\ &\quad \times \exp\left\{-\left(\frac{\mathcal{U}_v}{s}\right)^{-\alpha} - d_1\alpha - d_2s\right\} ds d\alpha \end{aligned} \quad (42)$$

Under the SEL function, the BPV of the future upper current record U_v^{c*} is the posterior predictive mean

$$\widehat{U}_v^{c*,BPV} = E(U_v^{c*} | \underline{U}^{c*}) = \int_{u_\mu}^{\infty} u_v f_{U_v^{c*} | \underline{U}^{c*}}^{\mathcal{P}}(u_v) du_v \quad (43)$$

Since the predictive density in (42) is also analytically intractable, the expectation in Eq. (43) does not admit a closed form expression. Accordingly, the Bayesian point predictor is approximated using Monte Carlo simulation:

$$\widehat{U}_v^{c*,BPV} \approx \frac{1}{k} \sum_{k=1}^k U_v^{c*(k)},$$

Where $\{U_v^{c*(k)}\}_{k=1}^k$ are draws from the posterior predictive^vdistribution obtained via MCMC. A $100(1 - \gamma)\%$ Bayesian predictive interval (L_γ, U_γ) for U_v^{c*} is defined by

$$P(L_\gamma \leq U_v^{c*} \leq U_\gamma | \underline{U}^{c*}) = 1 - \gamma$$

The corresponding bounds satisfy.

$$\int_{u_\mu}^{L_\gamma} f_{U_v^{c*} | \underline{U}^{c*}}^{\mathcal{P}}(u_v) du_v = \gamma/2, \quad (44)$$

$$\int_{u_\mu}^{U_\gamma} f_{U_v^{c*} | \underline{U}^{c*}}^{\mathcal{P}}(u_v) du_v = 1 - \gamma/2 \quad (45)$$

In practice, the Bayesian predictive interval is obtained from the ordered posterior predictive sample. $\{U_v^{c*(k)}\}_{k=1}^k$ as

$$(L_\gamma, U_\gamma) = (q_{\gamma/2}, q_{1-\gamma/2}).$$

6. Numerical Elucidation

6.1 Simulation Study

This section presents a comprehensive simulation study, along with the associated estimation procedures, for current record data under the Fréchet distribution. Simulation studies play a crucial role in scientific research, as they allow the performance of statistical methods to be systematically evaluated in controlled, reproducible settings. In particular, simulations enable the assessment of the finite-sample behavior of estimators, the examination of their bias, variability, and robustness, and the comparison of competing inferential approaches when analytical results are unavailable or difficult to obtain. In the context of record-based inference for the Fréchet distribution, simulation experiments provide valuable insight into the accuracy and stability of parameter estimates derived from incomplete data, such as current record values. Moreover, they help to validate theoretical findings, illustrate the practical feasibility of the proposed methods, and offer guidance on their implementation and expected performance in real-world applications.

Here we will generate 1000 samples with size 100, and in each sample, we will pick the upper and lower current records. Then we will estimate the parameters of each using MLE and BE techniques. After that, and for each sample of size μ , the mean of the estimates along with the standard deviation (SD) will be calculated, and the number of samples with that μ will be stored as ζ . In addition, the estimation accuracy was evaluated using the mean absolute deviation (MAD) and the root mean square error (RMSE). Furthermore, bootstrap confidence intervals (BCIs) were constructed for each model parameter. A comprehensive summary of these results is provided in Tables 1, 2, and 3

Table 1 (a): Results of Simulation when $\alpha = 2$ and $s = 1$

μ	Parameter	MLE			BE		
		Mean $\pm$ SD	RMSE/MAD	95% BCI	Mean $\pm$ SD	RMSE/MAD	95% BCI
4 ($\nu = 9$)	$\alpha_{U^{c*}}$	3.0066 $\pm$ 0.4281	1.0845 / 1.0066	[2.7447, 3.2673]	2.9861 $\pm$ 0.4397	1.0697 / 0.9861	[2.7010, 3.2545]
	$s_{U^{c*}}$	4.9559 $\pm$ 1.6584	4.2537 / 3.9559	[4.0131, 6.0124]	5.1943 $\pm$ 1.4268	4.4048 / 4.1943	[4.3586, 6.1221]
	$\alpha_{L^{c*}}$	2.8831 $\pm$ 0.6385	1.0688 / 0.8831	[2.4958, 3.2802]	2.9365 $\pm$ 0.7140	1.1533 / 1.0200	[2.4972, 3.3572]
	$s_{L^{c*}}$	0.5799 $\pm$ 0.3487	0.5334 / 0.4735	[0.3901, 0.7948]	0.5555 $\pm$ 0.3230	0.5388 / 0.4734	[0.3596, 0.7729]
5 ($\nu = 29$)	$\alpha_{U^{c*}}$	2.9749 $\pm$ 0.5779	1.1282 / 0.9749	[2.7505, 3.1788]	3.0482 $\pm$ 0.6571	1.2311 / 1.0482	[2.8113, 3.2998]
	$s_{U^{c*}}$	5.4657 $\pm$ 3.2455	5.4875 / 4.4657	[4.3745, 6.6141]	5.4133 $\pm$ 3.3123	5.4836 / 4.4133	[4.3199, 6.6173]
	$\alpha_{L^{c*}}$	3.0850 $\pm$ 0.5512	1.2127 / 1.0850	[2.8916, 3.2824]	2.9583 $\pm$ 0.6194	1.1353 / 0.9618	[2.7428, 3.1680]
	$s_{L^{c*}}$	0.7263 $\pm$ 0.5692	0.6226 / 0.5012	[0.5599, 0.9563]	0.7080 $\pm$ 0.5080	0.5783 / 0.4936	[0.5495, 0.9113]
6 ($\nu = 56$)	$\alpha_{U^{c*}}$	3.0810 $\pm$ 0.5612	1.2157 / 1.0810	[2.9377, 3.2120]	3.0942 $\pm$ 0.6346	1.2621 / 1.1009	[2.9373, 3.2654]
	$s_{U^{c*}}$	6.0565 $\pm$ 3.8115	6.3116 / 5.0565	[5.1150, 7.0164]	6.0835 $\pm$ 3.7928	6.3222 / 5.0835	[5.2587, 7.0818]
	$\alpha_{L^{c*}}$	3.0448 $\pm$ 0.5555	1.1810 / 1.0448	[2.9026, 3.1803]	3.0426 $\pm$ 0.6735	1.2379 / 1.0576	[2.8898, 3.2208]

Table 1 (b): Results of Simulation when $\alpha = 2$ and $s = 1$

μ	Parameter	MLE			BE		
		Mean $\pm$ SD	RMSE/MAD	95% BCI	Mean $\pm$ SD	RMSE/MAD	95% BCI
7 ($\nu = 109$)	S_{LC^*}	0.6780 $\pm$ 0.4726	0.5684 / 0.4896	[0.5646, 0.8124]	0.6503 $\pm$ 0.4475	0.5648 / 0.4977	[0.5427, 0.7747]
	α_{UC^*}	3.0268 $\pm$ 0.5844	1.1801 / 1.0268	[2.9156, 3.1367]	3.0074 $\pm$ 0.6382	1.1909 / 1.0146	[2.8922, 3.1318]
	S_{UC^*}	4.7679 $\pm$ 2.5679	4.5531 / 3.7679	[4.3169, 5.2384]	4.7262 $\pm$ 2.5947	4.5338 / 3.7262	[4.2598, 5.2379]
	α_{LC^*}	2.9983 $\pm$ 0.5831	1.1547 / 0.9983	[2.8944, 3.1120]	2.9565 $\pm$ 0.6844	1.1743 / 0.9758	[2.8315, 3.0876]
8 ($\nu = 151$)	S_{LC^*}	0.5640 $\pm$ 0.1998	0.4792 / 0.4523	[0.5259, 0.6015]	0.5741 $\pm$ 0.2298	0.4834 / 0.4546	[0.5308, 0.6169]
	α_{UC^*}	3.0265 $\pm$ 0.5704	1.1734 / 1.0265	[2.9357, 3.1240]	3.0577 $\pm$ 0.6519	1.2413 / 1.0651	[2.9662, 3.1566]
	S_{UC^*}	5.2982 $\pm$ 4.6794	6.3424 / 4.2982	[4.6724, 6.2816]	5.2717 $\pm$ 4.6782	6.3236 / 4.2717	[4.5732, 6.0515]
	α_{LC^*}	3.0349 $\pm$ 0.5964	1.1934 / 1.0349	[2.9363, 3.1310]	3.0418 $\pm$ 0.6524	1.2281 / 1.0640	[2.9476, 3.1601]
9 ($\nu = 179$)	S_{LC^*}	0.6048 $\pm$ 0.3038	0.4979 / 0.4608	[0.5602, 0.6522]	0.6110 $\pm$ 0.3084	0.4957 / 0.4555	[0.5666, 0.6685]
	α_{UC^*}	2.9457 $\pm$ 0.5734	1.1051 / 0.9457	[2.8712, 3.0352]	2.9508 $\pm$ 0.6107	1.1291 / 0.9601	[2.8630, 3.0494]
	S_{UC^*}	5.0600 $\pm$ 3.7325	5.5079 / 4.0600	[4.5660, 5.5387]	5.0662 $\pm$ 3.7290	5.5101 / 4.0662	[4.5573, 5.6255]
	α_{LC^*}	2.9895 $\pm$ 0.6142	1.1637 / 0.9895	[2.9096, 3.0800]	3.0536 $\pm$ 0.7055	1.2669 / 1.0682	[2.9537, 3.1611]
10 ($\nu = 135$)	S_{LC^*}	0.5489 $\pm$ 0.2696	0.5251 / 0.4829	[0.5188, 0.5917]	0.5504 $\pm$ 0.2685	0.5233 / 0.4789	[0.5165, 0.5884]
	α_{UC^*}	2.9221 $\pm$ 0.5721	1.0840 / 0.9221	[2.8217, 3.0218]	2.8795 $\pm$ 0.6361	1.0841 / 0.8891	[2.7826, 3.0018]
	S_{UC^*}	5.2548 $\pm$ 8.4823	9.4615 / 4.2548	[4.1057, 7.0965]	5.2573 $\pm$ 8.4843	9.4644 / 4.2573	[4.0155, 6.8800]
	α_{LC^*}	3.0097 $\pm$ 0.5999	1.1733 / 1.0097	[2.9135, 3.1006]	3.0287 $\pm$ 0.7330	1.2615 / 1.0474	[2.9014, 3.1605]
11 ($\nu = 142$)	S_{LC^*}	0.5538 $\pm$ 0.2775	0.5250 / 0.4888	[0.5165, 0.6100]	0.5680 $\pm$ 0.2984	0.5244 / 0.4768	[0.5282, 0.6279]
	α_{UC^*}	2.9627 $\pm$ 0.5673	1.1164 / 0.9627	[2.8713, 3.0588]	3.0084 $\pm$ 0.6847	1.2175 / 1.0273	[2.8968, 3.1247]
	S_{UC^*}	4.4248 $\pm$ 2.5763	4.2802 / 3.4248	[4.0213, 4.8642]	4.4125 $\pm$ 2.5808	4.2730 / 3.4125	[4.0079, 4.8519]
	α_{LC^*}	2.9187 $\pm$ 0.5667	1.0783 / 0.9187	[2.8307, 3.0158]	2.9526 $\pm$ 0.6839	1.1713 / 0.9702	[2.8453, 3.0630]
12 ($\nu = 97$)	S_{LC^*}	0.5416 $\pm$ 0.1936	0.4973 / 0.4767	[0.5113, 0.5751]	0.5478 $\pm$ 0.2083	0.4976 / 0.4719	[0.5183, 0.5853]
	α_{UC^*}	3.0089 $\pm$ 0.5534	1.1493 / 1.0089	[2.8934, 3.1226]	3.0156 $\pm$ 0.5911	1.1736 / 1.0245	[2.9060, 3.1298]
	S_{UC^*}	4.0584 $\pm$ 1.9947	3.6458 / 3.0584	[3.6681, 4.4380]	4.0230 $\pm$ 2.0040	3.6212 / 3.0230	[3.6427, 4.4248]
	α_{LC^*}	3.1109 $\pm$ 0.5798	1.2517 / 1.1109	[2.9854, 3.2215]	3.1714 $\pm$ 0.7144	1.3702 / 1.1785	[3.0298, 3.3223]
13 ($\nu = 45$)	S_{LC^*}	0.5203 $\pm$ 0.1098	0.4920 / 0.4797	[0.4998, 0.5413]	0.5241 $\pm$ 0.1179	0.4901 / 0.4759	[0.5011, 0.5496]
	α_{UC^*}	2.8181 $\pm$ 0.5024	0.9571 / 0.8181	[2.6750, 2.9568]	2.8124 $\pm$ 0.5415	0.9730 / 0.8170	[2.6606, 2.9617]
	S_{UC^*}	4.5769 $\pm$ 2.6102	4.4109 / 3.5769	[3.8925, 5.4401]	4.5024 $\pm$ 2.6348	4.3652 / 3.5024	[3.7707, 5.2985]
	α_{LC^*}	3.0369 $\pm$ 0.6308	1.2100 / 1.0369	[2.8584, 3.2241]	3.1073 $\pm$ 0.6641	1.2874 / 1.1274	[2.9031, 3.3137]
14 ($\nu = 17$)	S_{LC^*}	0.5225 $\pm$ 0.0961	0.4868 / 0.4775	[0.4945, 0.5499]	0.5194 $\pm$ 0.1090	0.4925 / 0.4806	[0.4849, 0.5498]
	α_{UC^*}	3.1088 $\pm$ 0.6125	1.2580 / 1.1088	[2.8379, 3.3837]	3.1284 $\pm$ 0.6434	1.2896 / 1.1284	[2.8711, 3.4441]
	S_{UC^*}	3.8027 $\pm$ 1.2959	3.0718 / 2.8027	[3.2399, 4.4437]	3.7846 $\pm$ 1.2559	3.0395 / 2.7846	[3.2902, 4.2924]
	α_{LC^*}	3.0162 $\pm$ 0.6495	1.1957 / 1.0162	[2.7038, 3.3021]	3.0726 $\pm$ 0.7658	1.3048 / 1.1052	[2.6483, 3.4086]
15 ($\nu = 14$)	S_{LC^*}	0.5141 $\pm$ 0.1093	0.4973 / 0.4859	[0.4660, 0.5673]	0.5289 $\pm$ 0.1528	0.4938 / 0.4711	[0.4681, 0.6106]
	α_{UC^*}	2.8070 $\pm$ 0.5725	0.9775 / 0.8070	[2.5449, 3.1373]	2.7378 $\pm$ 0.7511	1.0335 / 0.7378	[2.4063, 3.1471]
	S_{UC^*}	3.5722 $\pm$ 2.0195	3.2255 / 2.5722	[2.6944, 4.7692]	3.6612 $\pm$ 2.0009	3.2863 / 2.6612	[2.8255, 4.6915]
	α_{LC^*}	3.2148 $\pm$ 0.6317	1.3588 / 1.2148	[2.8921, 3.5069]	3.4437 $\pm$ 0.7532	1.6159 / 1.4437	[3.0536, 3.8202]
16 ($\nu = 9$)	S_{LC^*}	0.5334 $\pm$ 0.1213	0.4811 / 0.4666	[0.4803, 0.5956]	0.5789 $\pm$ 0.1505	0.4454 / 0.4239	[0.5166, 0.6568]
	α_{UC^*}	2.8330 $\pm$ 0.5608	0.9866 / 0.8330	[2.4999, 3.2307]	2.7944 $\pm$ 0.5432	0.9452 / 0.8345	[2.4504, 3.0899]
	S_{UC^*}	4.8049 $\pm$ 2.6535	4.5536 / 3.8049	[3.5105, 6.5927]	4.6480 $\pm$ 2.7137	4.4558 / 3.6480	[3.3849, 6.4855]
	α_{LC^*}	3.2306 $\pm$ 0.6321	1.3674 / 1.2306	[2.8513, 3.5838]	3.1853 $\pm$ 0.5208	1.2830 / 1.1853	[2.8371, 3.4973]
	S_{LC^*}	0.5235 $\pm$ 0.0752	0.4818 / 0.4765	[0.4765, 0.5723]	0.5163 $\pm$ 0.1012	0.4930 / 0.4837	[0.4594, 0.5795]

Table 2 (a): Results of Simulation when $\alpha = 3$ and $s = 1$

μ	Parameter	MLE			BE		
		Mean $\pm$ SD	RMSE/MAD	95% BCI	Mean $\pm$ SD	RMSE/MAD	95% BCI
4 ($\nu = 9$)	α_{UC^*}	3.0287 $\pm$ 0.5284	0.4990 / 0.4439	[2.7159, 3.3459]	2.9822 $\pm$ 0.8395	0.7917 / 0.6805	[2.4751, 3.4772]
	S_{UC^*}	2.9085 $\pm$ 1.7398	2.5166 / 1.9085	[1.9653, 4.1383]	2.8560 $\pm$ 1.6632	2.4297 / 1.8724	[1.9097, 3.9889]
	α_{LC^*}	3.1757 $\pm$ 0.6420	0.6302 / 0.5745	[2.7825, 3.5656]	3.1635 $\pm$ 0.6471	0.6316 / 0.5679	[2.7506, 3.4969]
	S_{LC^*}	0.6960 $\pm$ 0.4482	0.5206 / 0.4826	[0.4826, 1.0325]	0.6983 $\pm$ 0.4942	0.5551 / 0.5095	[0.4612, 1.0285]
5 ($\nu = 26$)	α_{UC^*}	3.1418 $\pm$ 0.6278	0.6317 / 0.5615	[2.9079, 3.3926]	3.1462 $\pm$ 0.8092	0.8068 / 0.6900	[2.8094, 3.4688]
	S_{UC^*}	2.7753 $\pm$ 1.3826	2.2338 / 1.7753	[2.2790, 3.3463]	2.7469 $\pm$ 1.4125	2.2294 / 1.7469	[2.2709, 3.2871]
	α_{LC^*}	3.0044 $\pm$ 0.5614	0.5505 / 0.4682	[2.8062, 3.2154]	3.0199 $\pm$ 0.5910	0.5799 / 0.4869	[2.7922, 3.2374]
	S_{LC^*}	0.6835 $\pm$ 0.2572	0.4047 / 0.3633	[0.5925, 0.7909]	0.6861 $\pm$ 0.2991	0.4296 / 0.3890	[0.5796, 0.8094]
6 ($\nu = 53$)	α_{UC^*}	3.1435 $\pm$ 0.6420	0.6519 / 0.5868	[2.9688, 3.3149]	3.1581 $\pm$ 0.7304	0.7406 / 0.5944	[2.9750, 3.3801]
	S_{UC^*}	2.5265 $\pm$ 0.7550	1.6998 / 1.5265	[2.3312, 2.7203]	2.5146 $\pm$ 0.7615	1.6920 / 1.5146	[2.3285, 2.7292]
	α_{LC^*}	3.0118 $\pm$ 0.5377	0.5328 / 0.4516	[2.8457, 3.1518]	3.1015 $\pm$ 0.6465	0.6484 / 0.5329	[2.9359, 3.2682]
	S_{LC^*}	0.6413 $\pm$ 0.1630	0.3934 / 0.3672	[0.5986, 0.6803]	0.6485 $\pm$ 0.1811	0.3947 / 0.3653	[0.5971, 0.6954]
7 ($\nu = 118$)	α_{UC^*}	3.0261 $\pm$ 0.5802	0.5783 / 0.4956	[2.9370, 3.1245]	3.0649 $\pm$ 0.7093	0.7092 / 0.6150	[2.9446, 3.1926]
	S_{UC^*}	2.5709 $\pm$ 1.3639	2.0766 / 1.5709	[2.3613, 2.8632]	2.5583 $\pm$ 1.3750	2.0744 / 1.5599	[2.2971, 2.8345]
	α_{LC^*}	3.0485 $\pm$ 0.5744	0.5740 / 0.5038	[2.9431, 3.1516]	3.0573 $\pm$ 0.6549	0.6546 / 0.5370	[2.9314, 3.1793]
	S_{LC^*}	0.6110 $\pm$ 0.1964	0.4354 / 0.4139	[0.5785, 0.6469]	0.6137 $\pm$ 0.2211	0.4447 / 0.4184	[0.5734, 0.6525]
8 ($\nu = 165$)	α_{UC^*}	3.0407 $\pm$ 0.5938	0.5934 / 0.5186	[2.9588, 3.1302]	3.0336 $\pm$ 0.6393	0.6383 / 0.5348	[2.9332, 3.1300]
	S_{UC^*}	2.3327 $\pm$ 0.7913	1.5487 / 1.3327	[2.2323, 2.4590]	2.3322 $\pm$ 0.7560	1.5306 / 1.3322	[2.2255, 2.4420]
	α_{LC^*}	3.0035 $\pm$ 0.5689	0.5672 / 0.4867	[2.9154, 3.0833]	3.0266 $\pm$ 0.6565	0.6550 / 0.5374	[2.9283, 3.1262]

Table 2 (b): Results of Simulation when $\alpha = 3$ and $s = 1$

μ	Parameter	MLE			BE		
		Mean $\pm$ SD	RMSE/MAD	95% BCI	Mean $\pm$ SD	RMSE/MAD	95% BCI
9 ($\nu = 177$)	$S_{L^{cs}}$	0.5976 $\pm$ 0.1236	0.4209 / 0.4031	[0.5784, 0.6159]	0.5916 $\pm$ 0.1314	0.4289 / 0.4099	[0.5726, 0.6130]
	$\alpha_{U^{cs}}$	2.8855 $\pm$ 0.5726	0.5824 / 0.5102	[2.7947, 2.9719]	2.9199 $\pm$ 0.6838	0.6866 / 0.5560	[2.8239, 3.0133]
	$S_{U^{cs}}$	2.1866 $\pm$ 0.7349	1.3946 / 1.1866	[2.0701, 2.2930]	2.2105 $\pm$ 0.7876	1.4429 / 1.2105	[2.0853, 2.3400]
	$\alpha_{L^{cs}}$	2.9252 $\pm$ 0.5924	0.5955 / 0.5081	[2.8435, 3.0157]	2.9805 $\pm$ 0.7362	0.7344 / 0.6130	[2.8776, 3.0915]
	$S_{L^{cs}}$	0.5712 $\pm$ 0.1100	0.4426 / 0.4288	[0.5549, 0.5872]	0.5812 $\pm$ 0.1342	0.4396 / 0.4188	[0.5591, 0.6029]
10 ($\nu = 144$)	$\alpha_{U^{cs}}$	3.0281 $\pm$ 0.5607	0.5594 / 0.4866	[2.9409, 3.1174]	3.0325 $\pm$ 0.6543	0.6528 / 0.5370	[2.9247, 3.1417]
	$S_{U^{cs}}$	2.5493 $\pm$ 1.4466	2.1162 / 1.5493	[2.3361, 2.7954]	2.5323 $\pm$ 1.4749	2.1232 / 1.5323	[2.3199, 2.7993]
	$\alpha_{L^{cs}}$	2.9668 $\pm$ 0.5779	0.5769 / 0.5061	[2.8660, 3.0589]	3.0115 $\pm$ 0.6728	0.6706 / 0.5558	[2.9146, 3.1280]
	$S_{L^{cs}}$	0.5878 $\pm$ 0.1053	0.4253 / 0.4128	[0.5709, 0.6054]	0.6019 $\pm$ 0.1314	0.4191 / 0.3988	[0.5789, 0.6229]
	$\alpha_{U^{cs}}$	3.0189 $\pm$ 0.5897	0.5876 / 0.5081	[2.9136, 3.1110]	3.0653 $\pm$ 0.7057	0.7059 / 0.5946	[2.9259, 3.1917]
11 ($\nu = 125$)	$S_{U^{cs}}$	2.1537 $\pm$ 0.7792	1.3905 / 1.1537	[2.0232, 2.2930]	2.1540 $\pm$ 0.7915	1.3975 / 1.1540	[2.0222, 2.2989]
	$\alpha_{L^{cs}}$	2.9173 $\pm$ 0.5700	0.5737 / 0.4965	[2.8204, 3.0119]	2.9649 $\pm$ 0.6578	0.6561 / 0.5423	[2.8470, 3.0713]
	$S_{L^{cs}}$	0.5600 $\pm$ 0.0875	0.4485 / 0.4400	[0.5442, 0.5746]	0.5597 $\pm$ 0.0933	0.4500 / 0.4403	[0.5435, 0.5747]
	$\alpha_{U^{cs}}$	3.0400 $\pm$ 0.5261	0.5244 / 0.4361	[2.9100, 3.1539]	3.0144 $\pm$ 0.6220	0.6183 / 0.4873	[2.8881, 3.1533]
	$S_{U^{cs}}$	2.3230 $\pm$ 1.5842	2.0565 / 1.3230	[2.0649, 2.7482]	2.3448 $\pm$ 1.6167	2.0952 / 1.3448	[2.0656, 2.7863]
12 ($\nu = 81$)	$\alpha_{L^{cs}}$	3.0229 $\pm$ 0.5782	0.5751 / 0.4991	[2.8971, 3.1415]	3.0743 $\pm$ 0.7084	0.7079 / 0.5624	[2.9258, 3.2242]
	$S_{L^{cs}}$	0.5899 $\pm$ 0.1029	0.4227 / 0.4101	[0.5689, 0.6140]	0.6020 $\pm$ 0.1185	0.4151 / 0.3980	[0.5760, 0.6290]
	$\alpha_{U^{cs}}$	3.1425 $\pm$ 0.5520	0.5634 / 0.4842	[2.9687, 3.3042]	3.1829 $\pm$ 0.6795	0.6955 / 0.5766	[2.9782, 3.3905]
	$S_{U^{cs}}$	2.2422 $\pm$ 0.6798	1.4120 / 1.2422	[2.0511, 2.4666]	2.2457 $\pm$ 0.6530	1.4027 / 1.2457	[2.0527, 2.4433]
	$\alpha_{L^{cs}}$	2.9500 $\pm$ 0.5969	0.5915 / 0.5174	[2.7603, 3.1418]	2.9474 $\pm$ 0.6054	0.6001 / 0.5021	[2.7832, 3.1377]
13 ($\nu = 40$)	$S_{L^{cs}}$	0.5826 $\pm$ 0.0708	0.4232 / 0.4174	[0.5619, 0.6044]	0.5814 $\pm$ 0.0941	0.4288 / 0.4186	[0.5533, 0.6083]
	$\alpha_{U^{cs}}$	3.2174 $\pm$ 0.5437	0.5783 / 0.5089	[3.0398, 3.3911]	3.2942 $\pm$ 0.5270	0.5970 / 0.5293	[3.1089, 3.4728]
	$S_{U^{cs}}$	2.1230 $\pm$ 0.5447	1.2448 / 1.1230	[1.9608, 2.3001]	2.0985 $\pm$ 0.5228	1.2134 / 1.0985	[1.9387, 2.2770]
	$\alpha_{L^{cs}}$	2.9477 $\pm$ 0.5352	0.5301 / 0.4521	[2.7736, 3.1021]	2.9255 $\pm$ 0.6410	0.6361 / 0.5221	[2.7355, 3.1433]
	$S_{L^{cs}}$	0.5675 $\pm$ 0.0819	0.4399 / 0.4325	[0.5414, 0.5951]	0.5590 $\pm$ 0.1075	0.4536 / 0.4410	[0.5275, 0.5981]
14 ($\nu = 35$)	$\alpha_{U^{cs}}$	3.0104 $\pm$ 0.4593	0.4448 / 0.3915	[2.7928, 3.2328]	3.0423 $\pm$ 0.5724	0.5558 / 0.4259	[2.7529, 3.3156]
	$S_{U^{cs}}$	2.1412 $\pm$ 0.4904	1.2361 / 1.1412	[1.9145, 2.3818]	2.2239 $\pm$ 0.5234	1.3247 / 1.2239	[2.0027, 2.4649]
	$\alpha_{L^{cs}}$	3.2097 $\pm$ 0.4899	0.5186 / 0.4239	[2.9942, 3.4171]	3.2610 $\pm$ 0.5434	0.5873 / 0.4554	[3.0118, 3.5168]
	$S_{L^{cs}}$	0.6075 $\pm$ 0.0816	0.4004 / 0.3925	[0.5735, 0.6510]	0.5992 $\pm$ 0.0916	0.4105 / 0.4008	[0.5544, 0.6434]

Table 3 (a): Results of Simulation when $\alpha = 3$ and $s = 2$

μ	Parameter	MLE			BE		
		Mean $\pm$ SD	RMSE/MAD	95% BCI	Mean $\pm$ SD	RMSE/MAD	95% BCI
4 ($\nu = 20$)	$\alpha_{U^{cs}}$	3.1628 $\pm$ 0.5871	0.5949 / 0.5472	[2.9217, 3.3962]	3.2162 $\pm$ 0.7193	0.7337 / 0.6761	[2.8673, 3.4544]
	$S_{U^{cs}}$	6.2900 $\pm$ 2.6145	4.9898 / 4.2900	[5.2710, 7.5025]	6.3218 $\pm$ 2.5908	5.0054 / 4.3218	[5.2892, 7.3031]
	$\alpha_{L^{cs}}$	3.1135 $\pm$ 0.6098	0.6051 / 0.5305	[2.8420, 3.3410]	3.1966 $\pm$ 0.7693	0.7752 / 0.6084	[2.8685, 3.5351]
	$S_{L^{cs}}$	1.6186 $\pm$ 0.8363	0.9000 / 0.7983	[1.2831, 1.9610]	1.6711 $\pm$ 0.8587	0.8993 / 0.7918	[1.3539, 2.0634]
	$\alpha_{U^{cs}}$	2.8682 $\pm$ 0.5425	0.5484 / 0.4852	[2.6773, 3.0640]	2.8735 $\pm$ 0.6102	0.6120 / 0.4974	[2.6248, 3.1113]
5 ($\nu = 27$)	$S_{U^{cs}}$	6.3119 $\pm$ 3.7988	5.6999 / 4.3119	[5.1875, 7.9232]	6.2146 $\pm$ 3.8268	5.6449 / 4.2146	[5.0149, 7.8620]
	$\alpha_{L^{cs}}$	2.9666 $\pm$ 0.5549	0.5455 / 0.4624	[2.7438, 3.1935]	2.9736 $\pm$ 0.6558	0.6441 / 0.5188	[2.7526, 3.2276]
	$S_{L^{cs}}$	1.5609 $\pm$ 0.7536	0.8601 / 0.7286	[1.3074, 1.8547]	1.6133 $\pm$ 0.7871	0.8638 / 0.7467	[1.3499, 1.9368]
	$\alpha_{U^{cs}}$	2.8414 $\pm$ 0.5955	0.6121 / 0.5373	[2.7055, 2.9778]	2.8101 $\pm$ 0.6325	0.6560 / 0.5529	[2.6567, 2.9559]
	$S_{U^{cs}}$	4.3891 $\pm$ 1.2556	2.6947 / 2.3891	[4.1065, 4.6805]	4.3770 $\pm$ 1.2998	2.7046 / 2.3777	[4.1142, 4.7077]
6 ($\nu = 69$)	$\alpha_{L^{cs}}$	3.1313 $\pm$ 0.5813	0.5919 / 0.5246	[2.9914, 3.2647]	3.2060 $\pm$ 0.6702	0.6965 / 0.5817	[3.0665, 3.3550]
	$S_{L^{cs}}$	1.2470 $\pm$ 0.3339	0.8227 / 0.7681	[1.1731, 1.3253]	1.2668 $\pm$ 0.3897	0.8290 / 0.7695	[1.1858, 1.3601]
	$\alpha_{U^{cs}}$	3.0255 $\pm$ 0.5995	0.5971 / 0.5187	[2.9183, 3.1307]	3.0025 $\pm$ 0.6784	0.6751 / 0.5757	[2.8663, 3.1311]
	$S_{U^{cs}}$	5.1288 $\pm$ 1.9830	3.6991 / 3.1288	[4.7381, 5.5223]	5.0838 $\pm$ 2.0080	3.6745 / 3.0838	[4.7797, 5.4781]
	$\alpha_{L^{cs}}$	3.0244 $\pm$ 0.5733	0.5710 / 0.4928	[2.9164, 3.1278]	3.0758 $\pm$ 0.7065	0.7071 / 0.5926	[2.9396, 3.2124]
7 ($\nu = 102$)	$S_{L^{cs}}$	1.2331 $\pm$ 0.3022	0.8238 / 0.7773	[1.1740, 1.2931]	1.2707 $\pm$ 0.3601	0.8126 / 0.7624	[1.2017, 1.3475]
	$\alpha_{U^{cs}}$	3.0166 $\pm$ 0.5831	0.5815 / 0.4993	[2.9158, 3.1087]	3.0577 $\pm$ 0.6644	0.6648 / 0.5419	[2.9502, 3.1634]
	$S_{U^{cs}}$	5.3413 $\pm$ 4.5215	5.6105 / 3.3413	[4.7137, 6.1831]	5.3118 $\pm$ 4.5311	5.6007 / 3.3118	[4.7071, 6.0645]
	$\alpha_{L^{cs}}$	2.9887 $\pm$ 0.5818	0.5800 / 0.5058	[2.8866, 3.0783]	3.0214 $\pm$ 0.6761	0.6743 / 0.5427	[2.9190, 3.1281]
	$S_{L^{cs}}$	1.1391 $\pm$ 0.2161	0.8875 / 0.8609	[1.1066, 1.1705]	1.1462 $\pm$ 0.2606	0.8925 / 0.8538	[1.1068, 1.1883]
8 ($\nu = 157$)	$\alpha_{U^{cs}}$	2.9430 $\pm$ 0.5565	0.5578 / 0.4701	[2.8698, 3.0244]	2.9353 $\pm$ 0.6562	0.6574 / 0.5452	[2.8365, 3.0204]
	$S_{U^{cs}}$	4.7601 $\pm$ 1.8778	3.3352 / 2.7601	[4.5013, 5.0213]	4.7358 $\pm$ 1.8904	3.3222 / 2.7367	[4.4768, 5.0457]
	$\alpha_{L^{cs}}$	2.9397 $\pm$ 0.5457	0.5474 / 0.4626	[2.8515, 3.0218]	3.0071 $\pm$ 0.6801	0.6781 / 0.5597	[2.9054, 3.1086]
	$S_{L^{cs}}$	1.1521 $\pm$ 0.2010	0.8713 / 0.8479	[1.1209, 1.1809]	1.1560 $\pm$ 0.2442	0.8784 / 0.8440	[1.1206, 1.1923]
	$\alpha_{U^{cs}}$	2.9862 $\pm$ 0.5822	0.5806 / 0.4994	[2.8938, 3.0692]	2.9854 $\pm$ 0.6524	0.6506 / 0.5377	[2.8853, 3.0786]
9 ($\nu = 172$)	$S_{U^{cs}}$	4.5690 $\pm$ 1.5246	2.9849 / 2.5690	[4.3411, 4.8073]	4.5077 $\pm$ 1.5556	2.9484 / 2.5093	[4.2883, 4.7602]
	$\alpha_{L^{cs}}$	2.9966 $\pm$ 0.6369	0.6350 / 0.5645	[2.9027, 3.0968]	3.0052 $\pm$ 0.6959	0.6938 / 0.5831	[2.9035, 3.1093]
	$S_{L^{cs}}$	1.1398 $\pm$ 0.2201	0.8878 / 0.8653	[1.1053, 1.1758]	1.1486 $\pm$ 0.2838	0.8972 / 0.8619	[1.1073, 1.1934]

Table 3 (b): Results of Simulation when $\alpha = 3$ and $s = 2$

μ	Parameter	MLE			BE		
		Mean $\pm$ SD	RMSE/MAD	95% BCI	Mean $\pm$ SD	RMSE/MAD	95% BCI
11 ($\nu = 131$)	$\alpha_{U^{c*}}$	3.0540 $\pm$ 0.5804	0.5807 / 0.5091	[2.9535, 3.1516]	3.0373 $\pm$ 0.6449	0.6435 / 0.5403	[2.9339, 3.1463]
	$S_{U^{c*}}$	4.4335 $\pm$ 1.6527	2.9381 / 2.4335	[4.1491, 4.7223]	4.4174 $\pm$ 1.6784	2.9392 / 2.4174	[4.1386, 4.7234]
	$\alpha_{L^{c*}}$	3.0426 $\pm$ 0.5242	0.5240 / 0.4451	[2.9507, 3.1281]	3.0613 $\pm$ 0.6633	0.6636 / 0.5435	[2.9492, 3.1750]
	$S_{L^{c*}}$	1.1687 $\pm$ 0.1987	0.8545 / 0.8337	[1.1351, 1.2013]	1.1907 $\pm$ 0.2571	0.8488 / 0.8167	[1.1482, 1.2375]
	$\alpha_{U^{c*}}$	3.0519 $\pm$ 0.6049	0.6027 / 0.5375	[2.9031, 3.1902]	3.0507 $\pm$ 0.6259	0.6234 / 0.5291	[2.9020, 3.1824]
12 ($\nu = 69$)	$S_{U^{c*}}$	4.8603 $\pm$ 5.5087	6.1715 / 2.8603	[4.0076, 6.3085]	4.8305 $\pm$ 5.5192	6.1670 / 2.8305	[3.9666, 6.3447]
	$\alpha_{L^{c*}}$	3.0408 $\pm$ 0.5344	0.5321 / 0.4579	[2.9196, 3.1662]	3.0739 $\pm$ 0.6799	0.6790 / 0.5424	[2.9277, 3.2213]
	$S_{L^{c*}}$	1.1640 $\pm$ 0.1696	0.8528 / 0.8360	[1.1282, 1.2025]	1.1421 $\pm$ 0.2128	0.8835 / 0.8579	[1.0878, 1.1940]
	$\alpha_{U^{c*}}$	3.0656 $\pm$ 0.5696	0.5667 / 0.4791	[2.8920, 3.2252]	3.1563 $\pm$ 0.6175	0.6300 / 0.5242	[2.9826, 3.3242]
	$S_{U^{c*}}$	4.1429 $\pm$ 0.8597	2.3052 / 2.1429	[3.8984, 4.4172]	4.1043 $\pm$ 0.9014	2.2851 / 2.1043	[3.7986, 4.3685]
13 ($\nu = 43$)	$\alpha_{L^{c*}}$	2.9986 $\pm$ 0.5736	0.5669 / 0.4845	[2.8312, 3.1765]	3.0729 $\pm$ 0.6508	0.6473 / 0.5532	[2.9143, 3.2609]
	$S_{L^{c*}}$	1.1826 $\pm$ 0.2084	0.8430 / 0.8174	[1.1191, 1.2440]	1.1697 $\pm$ 0.2373	0.8628 / 0.8303	[1.0990, 1.2372]
	$\alpha_{U^{c*}}$	3.1202 $\pm$ 0.6119	0.6115 / 0.5276	[2.9082, 3.3494]	3.0753 $\pm$ 0.5553	0.5492 / 0.4585	[2.8668, 3.2744]
	$S_{U^{c*}}$	4.2496 $\pm$ 1.0122	2.4586 / 2.2496	[3.8358, 4.6394]	4.3079 $\pm$ 1.0135	2.5125 / 2.3079	[3.9227, 4.6373]
	$\alpha_{L^{c*}}$	3.0283 $\pm$ 0.6775	0.6645 / 0.6037	[2.8099, 3.2734]	2.9615 $\pm$ 0.6538	0.6417 / 0.5536	[2.6956, 3.2138]
14 ($\nu = 25$)	$S_{L^{c*}}$	1.1120 $\pm$ 0.1680	0.9031 / 0.8880	[1.0476, 1.1775]	1.0964 $\pm$ 0.1613	0.9174 / 0.9036	[1.0360, 1.1579]
	$\alpha_{U^{c*}}$	2.7753 $\pm$ 0.4639	0.5023 / 0.3976	[2.5539, 2.9970]	2.7755 $\pm$ 0.5418	0.5706 / 0.4333	[2.5233, 3.0250]
	$S_{U^{c*}}$	4.2368 $\pm$ 1.0137	2.4427 / 2.2368	[3.7584, 4.7023]	4.1859 $\pm$ 0.9961	2.3892 / 2.1859	[3.6944, 4.6707]
	$\alpha_{L^{c*}}$	2.9839 $\pm$ 0.5896	0.5711 / 0.5142	[2.7159, 3.2798]	2.8983 $\pm$ 0.7363	0.7202 / 0.6052	[2.5795, 3.2565]
	$S_{L^{c*}}$	1.1488 $\pm$ 0.1656	0.8662 / 0.8512	[1.0711, 1.2387]	1.1284 $\pm$ 0.1578	0.8849 / 0.8716	[1.0551, 1.2026]

Tables 1–3 present an extensive simulation study evaluating the finite-sample performance of the MLE and BEs for the Fréchet distribution parameters based on upper and lower current records. Overall, the simulation design is sound, and the reported performance measures (Mean $\pm$ SD, RMSE, MAD, and bootstrap confidence intervals) are appropriate and sufficiently informative.

Across all parameter configurations, both estimation methods yield point estimates that are generally close to the true parameter values, indicating approximate unbiasedness. Small deviations are observed for very small numbers of current records, particularly for the shape parameter estimated from upper records, but these discrepancies decrease systematically as the number of records μ increases. This pattern suggests empirical consistency of both estimators.

The variability measures reported in the tables reveal a clear improvement in estimation accuracy with increasing μ . Standard deviations, RMSEs, and MADs decrease monotonically as more current records become available, reflecting the increasing information content of the record sequences. Estimation of the scale parameter exhibits greater variability than that of the shape parameter, especially when based on upper current records, consistent with the heavy-tailed nature of the Fréchet distribution.

A consistent feature across all tables is that inference based on lower current records is more stable than inference based on upper current records. Estimates derived from lower records typically exhibit smaller dispersion measures and narrower confidence intervals. This behavior is expected, as extreme observations more strongly influence upper records and therefore display greater sampling variability.

The comparison between MLE and BE does not reveal a uniformly superior method. MLEs often produce slightly smaller standard deviations and narrower confidence intervals, particularly for moderate to large values of μ . BEs, on the other hand, show comparable accuracy and, in some cases, lower MAD values, indicating greater robustness in sparse-data settings. As μ increases, differences between the two approaches become negligible.

Finally, comparisons across different scale parameter settings indicate that larger scale values lead to greater variability and wider confidence intervals, particularly for upper-record-based estimation. Nevertheless, both estimation methods remain well centered around the true values, demonstrating robustness of the proposed procedures to changes in model scale.

In summary, the simulation results provide strong empirical support for the validity and reliability of both maximum likelihood and Bayesian inference based on current records for the Fréchet distribution. The conclusions drawn from the tables are well supported by the numerical evidence reported.

6.2 Real Saudi Industrial Data

Within Saudi Arabia's Industrial Production Index (IPI), activities related to water supply, sewerage, waste management, and environmental remediation have gained prominence, reflecting the Kingdom's long-term emphasis on sustainable development and modern infrastructure. This category encompasses a wide range of essential services, including desalination operations, water transmission, and distribution systems, wastewater collection and treatment, solid waste handling, and remediation of environmental hazards, all of which are vital in a country facing chronic water scarcity.

Movements in the IPI for this sector are strongly influenced by public investment initiatives aligned with Vision 2030, demographic expansion, urbanisation, and the growing needs of industrial activities for reliable water and sanitation services. An upward trajectory in the index typically signals capacity expansion, technological upgrading, and efficiency gains. In contrast, short-term variations may arise from operational maintenance, seasonal consumption patterns, or adjustments in government spending. With the progressive strengthening of environmental policies and greater involvement of private entities, this sector is anticipated to play a more consistent and influential role in shaping assessments of Saudi Arabia's Industrial Output and Environmental Sustainability.

The following data are the monthly index for the Oil Activities from January 2023 to August 2025 obtained from the Saudi General Authority of Statistics (SGAS):

112.2, 111.3, 113.4, 113.1, 108.0, 107.9, 99.5, 98.4, 101.1, 95.4, 94.2, 97.5, 97.7, 100.0, 98.8, 98.8, 101.7, 96.5, 97.5, 100.3, 100.4, 100.2, 97.1, 98.2, 97.7, 99.2, 101.6, 100.3, 101.8, 103.3, 106.7, 108.6.

Other IPI data and ours can be found on this link: <https://www.stats.gov.sa/en/statisticstabs?tab=436312&category=123454>. After examining several candidate distributions to identify an appropriate fit for the data, the Fréchet distribution was found to provide the best fit. The estimated shape and scale parameters were $\hat{\alpha} = 26.6847$ and $\hat{s} = 99.3902$, respectively. The adequacy of this fit was assessed using standard goodness-of-fit procedures, including the Kolmogorov-Smirnov test ($p\text{-value} = 0.617524$), the Anderson-Darling test ($p\text{-value} = 0.568566$), and the Cramér-von Mises test ($p\text{-value} = 0.550721$). The large p -values obtained from all three tests indicate strong agreement between the proposed Fréchet model and the observed data. Figure 4 illustrates the fitted Fréchet probability density function and cumulative distribution function superimposed on the empirical data

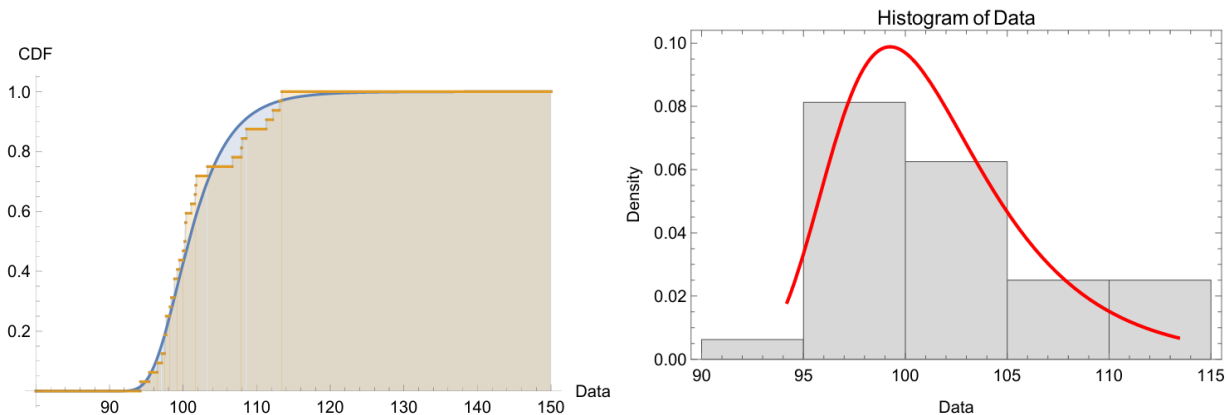


Figure 5: F (26.6847, 99.3902)'s CDF and PDF Against the Real Data

We now proceed to predict future upper and lower current records, as well as record ranges, using the two proposed approaches: the NBPI method and the BPV incorporating BPI.

First: NBPI

An algorithm was proposed in Subsection 5.1, which makes the NBPI very easy to apply.

We will run this algorithm here:

Step (1) already done with $\hat{\alpha} = 26.6847$ and $\hat{s} = 99.3902$ as an estimation for the parameters.

Step (2)

Table 4: Values of L_{μ}^{c*} , U_{μ}^{c*} and R_{μ}^{c*} from the Data

μ	1	2	3	4	5	6	7	8	9
U_{μ}^{c*}	112.2	112.2	113.4	113.4	113.4	113.4	113.4	113.4	113.4
L_{μ}^{c*}	112.2	111.3	111.3	108	107.9	99.5	98.4	95.4	94.2
R_{μ}^{c*}	0	0.9	2.1	5.4	5.5	13.94	15	18	19.2

Step (3) Since the simulation study in Subsection 6.1 shows that both BE and MLE perform well here, we will adopt the BE for parameter estimation. Applying the formulas from Subsection 4.2, we obtained that $\hat{\alpha}_{L^{c*}}^{BE} = 11.2506$, $\hat{s}_{L^{c*}}^{BE} = 104.2441$, $\hat{\alpha}_{U^{c*}}^{BE} = 14.7822$ and $\hat{s}_{U^{c*}}^{BE} = 87.5142$.

Step (4) To predict the next three upper and lower current records, we consider the cases $\nu = 1, 2$, and 3. For a significance level $\gamma = 0.05$ and $\mu = 9$, the corresponding quantiles $\Omega_{\nu\gamma}$ are $\Omega_{1,0.05} = 0.320292$, $\Omega_{2,0.05} = 0.526398$, and $\Omega_{3,0.05} = 0.716627$. The resulting NBPIs for the next three future current records are reported in Table 5.

Step (5)

Table 5: NBPI for the Next Three Current Records

$\mu = 9$	$U_{\mu+v}^{c*}$	$L_{\mu+v}^{c*}$	$R_{\mu+v}^{c*}$
$\nu = 1$	(113.400, 123.306)	(91.902, 94.200)	(19.200, 31.404)
$\nu = 2$	(113.400, 130.105)	(90.725, 94.200)	(19.200, 39.381)
$\nu = 3$	(113.400, 136.705)	(89.783, 94.200)	(19.200, 46.922)

Second: BPV and BPI

To predict future lower current records using the BPV approach with BPIs, equations 37, 38, and 39 are applied to the data presented in Table 4. A similar procedure is employed for upper current records using equations 42, 43, and 44. Predictions are then obtained for the next three current records, together with their corresponding BPIs, for $\nu = 1, 2$, and 3. The results are summarized in Table 6.

Table 6: BP and BPI for the Next Three Current Records

9	$\hat{U}_{\mu+v}^{c*}$	$\hat{L}_{\mu+v}^{c*}$	$\hat{R}_{\mu+v}^{c*}$
$\nu = 1$	125.969(113.66, 167.262)	89.4337(77.8834, 94.0605)	36.5353(35.7766, 73.2015)
$\nu = 2$	127.135(113.681, 173.833)	89.3607(81.0645, 93.4551)	37.7743(32.61657, 80.3779)
$\nu = 3$	127.673(113.678, 175.317)	88.8882(81.0434, 92.9578)	38.7848(32.6346, 82.3592)

A key feature of the Saudi IPI data is the extremely limited variability in the upper current record sequence. As shown in Table 4, the upper records take only two distinct values and remain constant at $U_{\mu}^{c*} = 113.4$ from $\mu = 3$ onward. This repetition leads to weak identifiability of the upper-record distribution and a flat likelihood surface, thereby inflating posterior uncertainty within the Bayesian framework. As a result, the Bayesian predictive intervals (BPIs) in Table 6 are excessively wide for future upper records and record ranges, reflecting posterior over-dispersion rather than genuine variability. Unlike the NBPI results in Table 5, Table 6 also provides a Bayesian point estimate for each predicted record, offering an expected value in addition to the interval.

In contrast, the lower current records exhibit more gradual variation, resulting in narrower and more informative BPIs in Table 6 than the corresponding intervals in Table 5. This is because the greater empirical variability in the lower records provides more information for estimation, reducing posterior uncertainty. The NBPI method in Table 5, while not producing point estimates, remains stable for both upper and lower records, as it enforces record constraints and is less sensitive to repeated values.

Overall, these observations illustrate that when upper records are largely constant, Bayesian intervals may overstate uncertainty, whereas NBPI provides coherent predictions. For lower records with greater observed variability, Bayesian prediction performs well and offers both point estimates and credible intervals, giving a more complete picture of expected future extremes.

7. Conclusion

This study has developed a comprehensive inferential framework for analyzing current record data from the Fréchet distribution, addressing both estimation and prediction challenges that arise when only record-breaking observations are available. The research contributes several key methodological advancements and practical insights to the statistical literature on record-based inference.

First, explicit closed-form expressions were derived for the cumulative distribution functions and probability density functions of both upper and lower current Fréchet records. These foundational results provide the mathematical basis for subsequent inference procedures and extend the theoretical understanding of record statistics under heavy-tailed distributions. The derived expressions reveal how the record-generating mechanism interacts with the Fréchet distribution's characteristic tail behavior, offering insight into the probabilistic structure of extreme sequential observations.

Second, moment expressions for current record values were obtained, thereby enabling the characterisation of central tendency and dispersion. These moment formulas are particularly valuable for applications that require quantitative summaries of record behavior, such as risk assessment and reliability analysis, where the expectations and variances of extreme values play crucial roles in decision-making.

Third, robust estimation procedures were developed using both maximum likelihood and Bayesian approaches. The simulation study demonstrated that both methods perform well across various parameter configurations, with Bayesian estimation showing particular strength in small- sample settings through its incorporation of prior information. The comparative analysis revealed that inferences based on lower current records generally exhibit greater stability than those based on upper records, reflecting the inherent variability of extreme upper observations under heavy- tailed distributions.

Fourth, two complementary prediction approaches were proposed: non-Bayesian prediction intervals based on distributional properties of record sequences, and Bayesian predictive intervals derived from posterior predictive distributions. These methods enable reliable forecasting of future record values and record ranges, with applications in monitoring, early warning systems, and performance assessment. The Saudi industrial data application illustrated the practical utility of these prediction methods in real-world settings, highlighting their relevance to industrial monitoring and reliability engineering.

The methodological developments presented in this work have several important implications. For statistical theory, they extend the record-based inference toolkit to encompass the Fréchet distribution, which is fundamental in extreme value analysis. For applied researchers, they provide concrete procedures for analyzing record data when complete samples are unavailable—a common scenario in reliability testing, environmental monitoring, and financial risk assessment. The integration of Bayesian methods offers particular advantages in settings where prior information is available or data limitations necessitate careful quantification of uncertainty.

Looking forward, several directions for future research emerge from this work. Extensions to more complex models, such as mixtures or composite Fréchet distributions, could enhance applicability to heterogeneous data settings. Developing sequential prediction methods that update as new records are observed would be valuable for real-time monitoring applications. Additionally, exploring the performance of record-based inference under various censoring schemes would broaden the practical scope of these methods. The application to Saudi industrial data also suggests opportunities for domain-specific methodological adaptations, particularly in energy and infrastructure sectors undergoing rapid transformation under Vision 2030.

In conclusion, this research provides a statistically rigorous yet practically accessible framework for current record analysis under the Fréchet distribution. By bridging theoretical developments with applied methodology, it offers valuable tools for researchers and practitioners working with extreme or record-breaking data across numerous scientific and engineering domains.

8. Acknowledgements

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/36482).

9. References

- Ahmadi, J., & Balakrishnan, N. (2005). Distribution-free confidence intervals for quantile intervals based on current records. *Statistics & probability letters*, 75(3), 190-202.
- AL-Dayian, G. R., EL-Helbawy, A. A., Hozaien, H., & Gawdat, E. (2024). Estimation of the Topp-Leone alpha power Weibull distribution based on lower record values. *Computational Journal of Mathematical and Statistical Sciences*, 3(1), 145-160.
- Aldallal, R. A. (2026). Bayesian estimation and prediction of Weibull current records with application to Saudi industrial data. *AIMS MATHEMATICS*, 11(2), 4369-4394.
- Alsanea, M. A. S. (2024). The role of record values in statistical inference: A review article. *Reliability: Theory & Applications*, 19(1 (77)), 406-423.
- Arnold, B. C., & Press, S. J. (1989). Bayesian estimation and prediction for Pareto data. *Journal of the American Statistical Association*, 84(408), 1079-1084.
- Barakat, H., Nigm, E., & Aldallal, R. (2014). Exact prediction intervals for future current records and record range from any continuous distribution. *SORT-Statistics and Operations Research Transactions*, 38(2), 251-270.
- Basak, P., & Balakrishnan, N. (2003). Maximum likelihood prediction of future record statistic. In *Mathematical and statistical methods in reliability* (pp. 159-175). World Scientific.
- Chacko, M., & Mary, M. S. (2013). Estimation and prediction based on k-record values from normal distribution. *Statistica*, 73(4), 505-516.
- Chahkandi, M., & Ahmadi, J. (2013). Prediction Intervals for k-records in Terms of Current Records. *Journal of Statistical Theory and Applications*, 12(1), 67-82.
- Einmahl, J. H., & Magnus, J. R. (2008). Records in athletics through extreme-value theory. *Journal of the American Statistical Association*, 103(484), 1382-1391.

- El-Din, M. M., Kotb, M., & Newer, H. (2015). Bayesian estimation and prediction for Pareto distribution based on ranked set sampling. *Journal of Statistics Applications & Probability*, 4(2), 211.
- Empacher, C., & Kamps, U. (2025). Simultaneous prediction of record values. *Metrika: International Journal for Theoretical and Applied Statistics*, 88(6), 1571-1589.
- Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., & Rubin, D. B. (2013). *Bayesian Data Analysis*. CRC Press.
- Gulati, S., & Padgett, W. J. (2003). *Parametric and nonparametric inference from record-breaking data* (Vol. 172). Springer Science & Business Media.
- Houchens, R. L. (1984). *Record value theory and inference*. University of California, Riverside.
- Kim, Y., & Seo, J. I. (2020). objective Bayesian prediction of future record statistics based on the exponentiated Gumbel distribution: Comparison with time-series prediction. *Symmetry*, 12(9), 1443.
- Muraleedharan, L., & Chacko, M. (2025). Joint Prediction of Future k-Record Values from a Generalized Exponential Distribution. *American Journal of Mathematical and Management Sciences*, 44(1), 112-129.
- Raqab, M. Z. (2009). Distribution-free prediction intervals for the future current record statistics. *Statistical Papers*, 50(2), 429-439.
- Sana, S., & Faizan, M. (2021). Bayesian estimation using lindley's approximation and prediction of generalized exponential distribution based on lower record values. *Journal of Statistics Applications & Probability*, 10(1), 61-75.
- Shafay, A. (2022). Bayesian Inference based on Pooled Sample from Two Independent Samples of Record Values. *Appl. Math*, 16(4), 601-610.
- Volovskiy, G., & Kamps, U. (2020). Maximum observed likelihood prediction of future record values. *Test*, 29(4), 1072-1097.