

Optimal Control and Dynamical Analysis of a Fractional-Order Infectious Disease Model with Behavioral Changes

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ABSTRACT

Mathematical models that capture time-memory and behavioral interventions within a single dynamical system can be used to investigate infectious disease dynamics. In this study, we provide a Caputo fractional-order SEIR model with preventive controls and behavioral awareness. The equilibrium and stability analysis show disease persistence and eradication, while the fractional approach captures nonlocal temporal dependence in the evolution of the model's state variables. We formulate the constrained optimization problem using Pontryagin's maximum principle and use normalized sensitivity analysis to assess the impact of fractional-order and epidemiological parameters. We further investigate the forward-backward sweep technique using numerical convergence analysis. Using the same parameterizations and initial conditions, we perform updated numerical simulations of the susceptible, infected, and recovered states under different fractional orders and controls. Within the same fractional-order system, the trajectory technique provides an analytical comparison of memory effects and behavior modulation. By combining optimal control, sensitivity analysis, convergence analysis, and fractional differential equation modeling, the paper provides a mathematical framework for understanding the dynamic behavior of infectious diseases.

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1. Introduction

A health department's policy decisions can be logically supported by formulating a disease in terms of mathematical models, which can aid in the prediction of disease transmission and control (Abdoon et al., 2023). To analyze and manage infectious disease spread, mathematical models are essential. One significant advancement in understanding how disease spreads has been the development of epidemic compartmental models by mathematicians and biologists worldwide. These models are useful tools for examining disease dynamics and supporting the development of effective preventive and therapeutic approaches (Sinha & Soni, 2025). In these models, infectious diseases are represented as dynamic transmission cycles involving interactions between susceptible and sick hosts. When analyzing infectious disease systems, the SIR model considers several factors. SIS, SIRV, SEIR, SVEIRS, SIQR, SEIRV, SEIQS, and other models are among those that look at important facets of the study of infectious diseases (Alfwzan et al., 2023).

Fractional-order epidemic models describe disease transmission using a nonlinear dynamical system in which the Caputo differential operator of order ($0 < \alpha \leq 1$) generates a non-local dependence on the system's history. As a result, the epidemic dynamics depend on both the state variables' values and their temporal memory. In this model, disease survival or extinction depends on equilibria, reproduction numbers, Jacobian eigenvalues, and asymptotic stability (ABU HAMMAD et al., 2025). The transmission model can also be transformed into a constrained optimal control problem when it incorporates behavioral awareness, prevention, treatment, and recovery controls. In this case, the model selects feasible control functions to optimize disease burden and intervention costs. Therefore, optimal control theory, stability analysis, sensitivity analysis, and fractional differential equations provide a robust mathematical framework for investigating the relationship between memory and behavioral intervention in infectious disease dynamics (Bouissa et al., 2023; Rezapour & Mohammadi, 2020). In control engineering, physics, neural networks, and medicine, fractional-order models have become increasingly popular. By accounting for a system's current state and all previous states, they can forecast its condition at any future point. Additionally, the fractional derivative is global, which makes it very helpful for modeling epidemic situations, whereas the integer derivative is local. Furthermore, the addition of a new parameter in fractional-

order systems significantly enhances numerical simulations (Odibat, 2024).

Combining optimal control theory with non-local fractional differential equations is a mathematically sound strategy to minimize disease transmission by accounting for memory-dependent biological mechanisms, as demonstrated by developments in fractional epidemiological modeling (Chauhan, 2025). Fractional-order epidemic models use non-local operators to preserve hereditary traits, whereas integer-order models cannot capture hereditary aspects because they rely on local operators (Naik, 2020). Additionally, Pontryagin's Maximum Principle, fractional Hamiltonian systems, adjoint equations, and cost-function optimization can determine the best vaccine, treatment, quarantine, and awareness strategies. Numerical simulations using predictor-corrector and Adams-type fractional methods show that control strategies employing fractional optimal control methods outperform their integer-order counterparts in optimizing the trade-off between intervention cost and epidemiological impact (Zabidi et al., 2022). Thus, optimal control and fractional calculus form a potent mathematical paradigm for evidence-based public health decision-making (Bolatova et al., 2025).

Fractional epidemic models have advanced significantly, but several mathematical issues remain unresolved. Fractional optimal control has not been used to examine how human behavior is affected in this environment; instead, most research to date has focused on disease transmission or therapies alone. Furthermore, these factors' combined effect on disease eradication has received little attention. By developing a fractional-order SEIR model with behavioral dynamics and an optimal control strategy, this study addresses these shortcomings. The framework uses Caputo fractional derivatives, applies Pontryagin's Maximum Principle to derive the optimality conditions, and numerically approximates the resulting state-adjoint system to determine the best intervention strategies. Beyond reducing infections and intervention costs, this integrated mathematical approach provides a more accurate explanation of epidemic spread. Thus, the suggested methodology offers a computationally efficient, mathematically rigorous framework aligned with the goals of optimal disease control and modern biomathematical modeling (Umarkar et al., 2025).

2. Research Objective of Study

The SEIR model captures memory effects in infectious disease dynamics by incorporating controls for awareness, prevention, treatment, and recovery. The study also considers the reproduction number, parameter sensitivity, optimal control, numerical convergence, equilibrium stability, and control efficiency.

3. Literature Review

To manage infected individuals, Kada et al. (2020) employed two control policies: mask use to protect susceptible body parts and hospital and quarantine isolation. According to their research, an all-encompassing approach that accounts for every control variable the World Health Organization (WHO) recommends produces exceptional results consistent with success. Harris and Bodmann (2022) suggest four control mechanisms to create a multi-region discrete mathematical model. This model captures the dynamics of new coronavirus transmission, including human-animal interactions within a single region or across multiple regions. Damak (2023) developed an optimal control problem related to Stuxnet virus dynamics. Using a modified Caputo-type fractional derivative, they examine the use of fractional derivatives in computer science. Further studies show that Hussain et al. (2021) studied an optimal control problem focused on pine wilt disease, building the model by parameterizing diseased pine trees in Korea between 2010 and 2019. Wang et al. (2022) examined the delay-dependent robust stability of uncertain fractional-order neutral-type systems with distributed delays and input saturation. They proposed state-feedback controller gains using linear matrix inequalities and verified the results with numerical simulations. To better represent disease dynamics, especially those with long incubation periods or complex interpersonal relationships, their mathematical epidemic models employed fractional-derivative results. Further studies show that fractional models improve understanding and forecasting of the epidemic's evolution over time. This enables researchers to better examine short-term variations, long-term trends, and the impact of historical conditions on current disease dynamics (Keeling et al., 2021). Several studies, including Feng et al. (2026), developed a framework for fractional-order epidemics that accounts for the optimal intervention strategy. They also established stability criteria using the fractional differential operator and used fixed-point theory to demonstrate the existence and uniqueness of the solution. The findings showed that memory-dependent transmission dynamics outperform classical ones in forecasting the course of an epidemic. Adu et al. (2024) developed a Caputo fractional SEIR model. They studied disease persistence using the basic reproduction number (R_0) technique and concepts of local and global stability. Their numerical investigations revealed that fractional-order models provide greater flexibility for non-linear infection dynamics and long-term epidemiological memory. The research reports the development of a generalized fractional epidemiological model that uses non-local operators to capture heterogeneous disease propagation. The combination of fractional calculus, existence theory, and nonlinear dynamics, according to the research, ensures that the resulting epidemic trajectories are more realistic because they retain characteristics absent in traditional ODEs (Pan et al., 2021). Further investigation shows that El-Mesady et al.

(2025) used Hamiltonian optimization in a fractional environment and Pontryagin's Maximum Principle to solve an optimal control problem of the spread of infectious diseases. They found that time-varying control strategies were inexpensive and effective at lowering infection rates. Research demo Ahmad et al. (2023) used high-order numerical approximation, sensitivity analysis, and Lyapunov stability theory to analyze the nonlinear fractional epidemic model. This study demonstrated the significance of the fractional order parameter in epidemic dynamics, convergence, and equilibrium stabilization. Few studies incorporate all three by accounting for behavioral response and time memory in the mathematical model. In contrast, previous works have examined fractional epidemic dynamics, stability analysis, and optimal control individually. Previous studies often use reproduction numbers to examine epidemic thresholds, Jacobian or Lyapunov approaches to analyze stability, and optimization control models to determine the best course of action. However, a single fractional SEIR model might not have all of the aforementioned elements. Therefore, the present study integrates the stability of the equilibrium point, behavior-dependent reproduction threshold value, sensitivity analysis, and Pontryagin optimality requirements with the fractional state equations. This method enables quantitative investigation of the impact of transmission rates, behavioral controls, and the order of the fractional derivatives on the dynamic system, including the system's trajectory and stability, as well as the efficacy of control measures.

4. Methodology

4.1. Preliminaries

Consider $L_1(t_0, t_f)$ be a Lebesgue space with $\alpha \geq 0$. The operator I_a^α is defined on $L_1(t_0, t_f)$ by,

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0 \quad (1)$$

$I_a^\alpha f(t) = f(t)$ is called the Riemann-Liouville fractional integral operator of order α , where $\Gamma(\cdot)$ is the Gamma function. The definition of caputo derivative is as:

$${}_t^C D_{0+}^\alpha f(t) = (I_{0+}^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \quad (2)$$

where $n-1 \leq \alpha < n, n \in \mathbb{N}, D = \frac{d}{dt}$ For the Caputo derivative, we get

$$\begin{cases} {}_t^C D_{0+}^\alpha (\beta f(x) + \gamma(g)) = \beta {}_t^C D_{0+}^\alpha f(x) + \gamma {}_t^C D_{0+}^\alpha g(x) \\ {}_t^C D_{0+}^\alpha \beta = 0 \end{cases} \quad (3)$$

where β and γ are constants for the Caputo derivative. The formula for the Caputo derivative of x^β is

$${}_t^C D_{0+}^\alpha x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < (\alpha) \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq (\alpha), \quad \beta > (\alpha) \end{cases} \quad (4)$$

where $\mathbb{N}_0 = (0, 1, 2, \dots)$, the ceiling function $\lceil \alpha \rceil$ is used to indicate the smallest integer larger than or equal to α , and the floor function $\lfloor \alpha \rfloor$ is used to indicate the greatest integer less than or equal to α .

4.2. Mathematical Model of Infectious Disease

The model separates the entire host population, $N(t)$, into four classes: susceptible ($S(t)$), exposed ($E(t)$), infected ($I(t)$), and recovered ($R(t)$). Let's now introduce the traditional SEIR mathematical model for infectious disease transmission. We believe that a person develops lifetime immunity after recovering from the disease. Furthermore, it is assumed that the birth and death rates, represented by ν , are equal and steady across time. As a result, the population size $N(t)$ stays constant over an extended period of time. In particular, $N(t) = S(t) + E(t) + I(t) + R(t)$. The following is model of the infectious disease transmission:

$$\begin{cases} \dot{S}(t) = \nu N - \rho S(t)I(t) - (\nu + \mu)S(t) \\ \dot{E}(t) = \rho S(t)I(t) - (\delta + \nu)E(t) \\ \dot{I}(t) = \delta E(t) - (\nu + \lambda + \beta + \gamma)I(t) \\ \dot{R}(t) = (\beta + \gamma)I(t) + \mu S(t) - \nu R(t) \end{cases} \quad (5)$$

Following $(S(0), E(0), I(0), R(0)) \in \mathbb{R}_+^4$ are now the initial conditions. Every parameter in these equations is non-negative.

At keep things simple, we set the population size at 1. Then, $s = \frac{S}{N}, e = \frac{E}{N}, i = \frac{I}{N}, r = \frac{R}{N}$ indicate the population's

proportions of susceptible, exposed, infectious, and recovered people, respectively, and meet the requirement that $s + e + i + r = 1$.

$$\begin{cases} \dot{s}(t) = \nu - \rho s(t)i(t) - (\nu + \mu)s(t) \\ \dot{e}(t) = \rho s(t)i(t) - (\delta + \nu)e(t) \\ \dot{i}(t) = \delta e(t) - (\nu + \lambda + \beta + \gamma)i(t) \\ \dot{r}(t) = (\beta + \gamma)i(t) + \mu s(t) - \nu r(t) \end{cases} \quad (6)$$

4.3. Fractional Order Model for Infectious Disease

First, we replaced each conventional derivative in the system with a Caputo fractional derivative of order α , where α is any real number between 0 and 1. Consequently, the following is the suggested fractional infectious model:

$$\begin{cases} {}^{C_0}D_0^\alpha s(t) = \nu - \rho s(t)i(t) - (\nu + \mu)s(t) \\ {}^{C_0}D_0^\alpha e(t) = \rho s(t)i(t) - (\delta + \nu)e(t) \\ {}^{C_0}D_0^\alpha i(t) = \delta e(t) - (\nu + \lambda + \beta + \gamma)i(t) \\ {}^{C_0}D_0^\alpha r(t) = (\beta + \gamma)i(t) + \mu s(t) - \nu r(t) \end{cases} \quad (7)$$

$(s(0), e(0), i(0), r(0)) \in R_+^4$ are the initial conditions. At $t \geq 0$, we assumed that $s(t), e(t), i(t), r(t)$ and their Caputo fractional derivatives are continuous functions. Without sacrificing generality, we examined the following system since $r(t)$ is absent from the first three equations in system (7):

$$\begin{cases} {}^{C_0}D_0^\alpha s(t) = \nu - \rho s(t)i(t) - (\nu + \mu)s(t) \\ {}^{C_0}D_0^\alpha e(t) = \rho s(t)i(t) - (\delta + \nu)e(t) \\ {}^{C_0}D_0^\alpha i(t) = \delta e(t) - (\nu + \lambda + \beta + \gamma)i(t) \end{cases} \quad (8)$$

$(s(0), e(0), i(0)) \in R_+^3$ as initial conditions. Furthermore, we have $r(t) = 1 - s(t) - e(t) - i(t)$ at any time t because the population is constant. The resulting fractional system uses the Caputo operator to simulate a nonlinear dynamical system whose state trajectories evolve in the feasible epidemiological region. The system's equilibrium and stability reflect the asymptotic behavior of disease transmission. At the same time, the boundedness and invariance of the state variables provide the mathematical foundation for evaluating the viability of the model's biological interpretation.

4.4. Dynamical Behavior of Fractional Order Model

We examined the following initial value issues for fractional differential equations in the form of in order to look into the existence, uniqueness, non-negativity of the solution, and stability of the equilibrium points of the system (8).

$$\begin{cases} {}^{C_0}D_0^\alpha X(t) = f(t, X(t)) \\ X(t_0) = X_0 \end{cases} \quad (9)$$

where $f(t, X(t)): R \times R^n \rightarrow R^n$ is a vector field. Let $J = [t_0 - a, t_0 + a]$, $B = [x \in R^n | x - x_0| \leq b]$, and $A = [(t, X) \in R \times R^n | t \in J, x \in B]$. Assume that the function meets the following requirements. $f: A \rightarrow R^n$. The initial value issue (9) has at least one solution on the interval $[t_0 - h, t_0 + h]$ for a positive number h if $\alpha > \frac{1}{2}$. Assume that the first two requirements are globally satisfied by the vector field. $f(t, X)$ and

$$\|f(t, X)\| \leq \omega + \theta \|X\| \quad (10)$$

For nearly all $t \in R$ and all $X \in R^n$. In this case, two positive constants are ω and θ . The problem (9) can therefore be solved by a function. $X(t)$ on $(-\infty, +\infty)$. Consider the equation (9) commensurate fractional-order system:

and $\alpha f(t, X(t)): R \times R^n \rightarrow R^n$ is a vector field and $\alpha \in (0, 1)$. Since the eigenvalue λ_i criterion for the linearized fractional order system ensures the equilibrium's local asymptotic stability, the local dynamics are mathematically described by the spectral character of the Jacobian matrix $\frac{\partial f(t, X)}{\partial X}$.

$$|\arg \lambda_i| > \alpha \frac{\pi}{2} \quad (11)$$

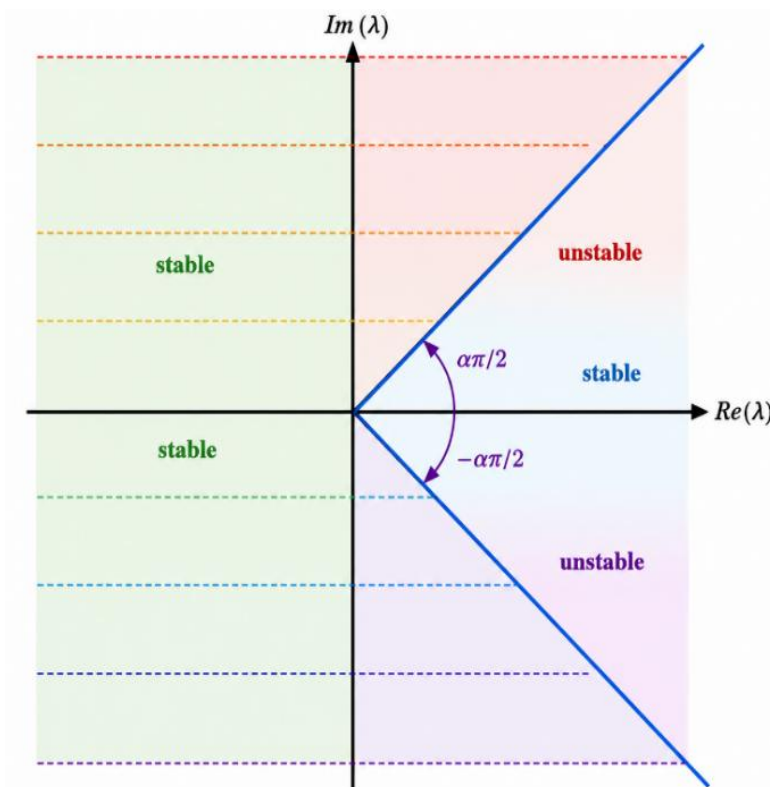


Figure 1: Stability Region of Fractional Order System

If and only if $f(t, X^*) = 0$, a point X^* is an equilibrium point of the equation (9) and it has the same equilibrium points as the integer-order system when $0 < \alpha < 1$. The model equation (8) has two equilibrium points:

- A disease-free equilibrium $E_f = \left(\frac{v}{v+\mu}, 0, 0\right)$.
- An infected steady-state equilibrium $E_e = (s^*, e^*, i^*)$, where

$$\begin{cases} s^* = \frac{v+\lambda+\beta+\gamma}{\rho\delta} \\ e^* = \frac{(\rho\delta v) - (v+\lambda+\beta+\gamma)(v+\mu)}{\rho\delta} \\ i^* = \frac{(\rho\delta v) - (v+\lambda+\beta+\gamma)(v+\mu)(v+\delta)}{\rho(v+\lambda+\beta+\gamma)(v+\delta)} \end{cases} \quad (12)$$

Let M be a real matrix with dimensions of 3×3 . Every eigenvalue of M has a negative real part if $tr(M)$, $det(M)$ and $det(M^{(2)})$ are all negative. The equilibrium E_f of the system (9) is locally asymptotically stable if,

$$\frac{(\rho\delta v)}{(v+\lambda+\beta+\gamma)(v+\mu)(v+\delta)} < 1 \quad (13)$$

We calculated the fractional system (8)'s Jacobian matrix at E_f :

$$J = \begin{bmatrix} -(v+\mu) & 0 & -\frac{\rho v}{v+\mu} \\ 0 & -(\delta+v) & \frac{\rho v}{v+\mu} \\ 0 & \delta & -(v+\lambda+\beta+\gamma) \end{bmatrix} \quad (14)$$

Then,

$$\begin{cases} tr(J) = -((v+\mu) + (\delta+v) + (v+\lambda+\beta+\gamma)) < 0 \\ \det(J) = -\left((v+\mu) + (\delta+v) + (v+\lambda+\beta+\gamma) - \frac{\rho v}{v+\mu}\right) < 0 \end{cases} \quad (15)$$

The Jacobian matrix's second compound is

$$J^2 = \begin{bmatrix} -(2v + \mu + \delta) & \frac{\rho v}{v + \mu} & -\frac{\rho v}{v + \mu} \\ \delta & -(2v + \mu + \lambda + \beta + \gamma) & 0 \\ 0 & 0 & -(\delta + 2v + \lambda + \beta + \gamma) \end{bmatrix} \quad (16)$$

J^2 's determinant is

$$\det(J^2) = -(\delta + 2v + \lambda + \beta + \gamma) \left((2v + \mu + \delta)(2v + \mu + \lambda + \beta + \gamma) - \frac{\rho v \delta}{v + \mu} \right) < 0 \quad (17)$$

Therefore, the equation (9)'s disease-free equilibrium at E_f is locally asymptotically stable if

$$\frac{(\rho \delta v)}{(v + \lambda + \beta + \gamma)(v + \mu)(v + \delta)} < 1 \quad (18)$$

The following lemma for fractional-order systems had to be introduced to use the Lyapunov functional to prove the global stability of the infected steady-state E_e .

4.5. Fractional Reproduction Number under Behavioral Intervention

A behaviorally dependent fractional reproduction number characterizes the transmission model's dynamics and accounts for the effects of the best control measures. When behavioral adaptation that influences the kinetics of transmission is taken into account, the infection rate function becomes

$$\beta_e(t) = \beta(1 - u_1(t))(1 - u_2(t)) \quad (19)$$

where, the awareness control is represented by $u_1(t)$, whereas $u_2(t)$ is the behavioral control for prevention. The definition of the effective basic reproduction number is

$$R_e = \frac{\beta_e \sigma}{(\sigma + \mu)(\gamma + \mu)} \quad (20)$$

In any case, disease-free equilibrium is epidemiologically achievable $R_e < 1$ and an endemic state still exists $R_e > 1$. The dependence of R_e on the stability and persistence of the disease-free and endemic equilibria is related to the transmission-function parameterization through a quantitative change in the epidemic threshold via behavioral restrictions. While the fundamental reproduction number threshold remains in the same algebraic form obtained using the next-generation approach, the fractional order affects the epidemic model's time dynamics and stability. The current formulation thereby captures the quantitative link between behavioral dynamics, fractional calculus, and epidemic management. This modeling approach allows evaluation of how public awareness and preventive actions affect disease elimination.

4.6. Fractional Optimal Control Formulation and Necessary Conditions

The following three limited and quantifiable controls, behavioral awareness, preventive behavior, and intervention for treatment and recovery, respectively, are then used to create the intervention problem: $u_1(t)$, $u_2(t)$, and $u_3(t)$. The expression for the admissible control set is

$$u = \left((u_1, u_2, u_3) : 0 \leq u_j(t) \leq 1, j = 1, 2, 3, t \in [0, T] \right) \quad (21)$$

The controlled transmission rate is characterized by

$$\beta_c(t) = \beta(1 - u_1(t))(1 - u_2(t)) \quad (22)$$

The appropriate recovery and treatment rate in the infected and recovered classes is used to introduce the treatment and recovery approach. Thus, the controlled fractional system can be expressed as

$${}^C D_{0+}^\alpha = F(X(t), u(t)), X(0) = X_0 \quad (23)$$

where,

$$X(t) = \begin{pmatrix} S(t) \\ E(t) \\ I(t) \\ R(t) \end{pmatrix}, u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \quad (24)$$

The objective is to reduce both the infectious load and the cost of implementing each of the three interventions. On the other hand, the performance index is provided by

$$J(u) = \int_0^T \left(A_I I(t) + \frac{B_1}{2} u_1^2(t) + \frac{B_2}{2} u_2^2(t) + \frac{B_3}{2} u_3^2(t) \right) dt \quad (25)$$

where B_1 , B_2 , and B_3 represent the relative implementation costs of the three controls, and A_I represents the epidemiological cost of infection. We obtain the Hamiltonian by applying Pontryagin's maximal principle.

$$H = \left(A_I I + \frac{B_1}{2} u_1^2, \frac{B_2}{2} u_2^2, \frac{B_3}{2} u_3^2 + \lambda^T F(X, u) \right) \quad (26)$$

In this case, the adjoint variables vector is denoted by λ . The ideal controls must meet the stationarity requirements.

$$\frac{\partial H}{\partial u_j} = 0, \quad j = 1, 2, 3 \quad (27)$$

in addition to the requirements for admission

$$0 \leq u_j^*(t) \leq 1 \quad (28)$$

Therefore, each optimal control is calculated by projecting its value on the permissible range $[0, 1]$ in the absence of these restrictions. In addition to optimizing interventions, the proposed modeling technique offers a unified method of mathematical modeling of memory-based epidemiological dynamics and behavior adaptability. Unlike conventional integer-order models, the current method optimizes awareness, preventive behavior, and treatment or recovery interventions simultaneously while accounting for the Caputo memory operator. As a result, the same optimum control system can be used to assess the impact of behavior adaptation and past illness dynamics.

4.7. Fractional Sensitivity Analysis of Model Parameters

We apply a normalized sensitivity analysis technique to the fractional model to evaluate the importance of the epidemiological factors. Every parameter

$$p \in (\beta, \sigma, \gamma, \mu, \alpha) \quad (29)$$

Here's how to calculate the normalized sensitivity index:

$$\gamma_p = \frac{\partial R_e}{\partial p} \times \frac{p}{R_e} \quad (30)$$

The sign of the sensitivity coefficient indicates whether the parameter affects the disease's dynamics. If $\gamma_p > 0$, epidemic spread is caused by an increase in the associated parameter. Conversely, $\gamma_p < 0$ it implies that raising the value lowers the spread of illness.

$$0 < \alpha \leq 1 \quad (31)$$

For the fractional-order situation, the parameter influences both equilibrium stability and transient behavior. Unlike sensitivity analysis, this model evaluates parameter impacts while accounting for genetic memory effects, allowing us to determine which mechanisms most affect an epidemic's persistence. This prioritization supports the biological significance of the optimal control model and aids the mathematical search for the best control strategy.

4.8. Convergence of the Fractional Optimal Control Algorithm

The convergence of the forward-backward sweep method is considered to assess the numerical soundness of the optimal control system. For this reason,

$$X^k = (S^k, E^k, I^k, R^k) \quad (32)$$

is the k^{th} iteration's state vector, and

$$U^k = (u_1^k, u_2^k, u_3^k) \quad (33)$$

It is the corresponding control vector. Convergence verification improves the model's numerical stability and computational reproducibility.

$$\varepsilon^k = \|X^{k+1} - X^k\| + \|U^{k+1} - U^k\| \quad (34)$$

The iterations are terminated whenever $\varepsilon^k < 10^{-6}$, the objective of functional also satisfies

$$J(U^{k+1}) \leq J(U^k) \quad (35)$$

the property that guaranties monotone behavior in the optimization process. Since the state equations are integrated forward and the adjoint equations backward based on Pontryagin's Maximum Principle, the iterations converge to an

admissible optimal solution that satisfies the necessary optimality conditions.

4.9. Fractional Control Efficiency Index

A fractional control efficiency index is defined to assess the effectiveness of the three control measures combined:

$$\eta = 1 - \frac{\int_0^T I_c(t) dt}{\int_0^T I_u(t) dt} \quad (36)$$

with $I_c(t)$ and $I_u(t)$ stands for the number of infected people with optimal control and those without optimal control. The index complies $0 \leq \eta \leq 1$. Low levels indicate insufficient effectiveness in containing the epidemic, whereas high values indicate an effective control plan. Unlike standard epidemic metrics that do not account for these aspects, the efficiency value integrates the effects of fractional memory, behavioral adaptation, and optimal control.

5. Results

5.1. Fractional-Order and Control-Dependent Susceptible Dynamics

For $\alpha = 0.75, 0.85, 0.95$, and 1.00 , the value of α was changed, all other parameters remained at the baseline levels. The values in Table 1 below were calculated using the revised baseline and chosen from the susceptible trajectory. The sensitivity to the fractional order parameter is shown in Table 1. In Table 2, numerical simulations compare no control, awareness and prevention, recovery or treatment, and a combination of the two controls under the assumption that $\alpha = 0.90$; this set is sensitive to various control scenarios. The values originate from the same fractional state system and initial condition, and they are all ratios of the normalized population.

Table 1: Proportion Susceptible to Changing α Values

t	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 1.00$
0	0.9300	0.9300	0.9300	0.9300
5	0.6618	0.6254	0.5865	0.5662
10	0.5072	0.4304	0.3485	0.3061
20	0.3338	0.2434	0.1662	0.1335
30	0.2626	0.1927	0.1455	0.1301
40	0.2330	0.1821	0.1554	0.1501
50	0.2202	0.1827	0.1681	0.1679
60	0.2145	0.1862	0.1785	0.1802

Table 2: Susceptible Proportion under Other Control Strategies ($\alpha = 0.90$)

t	No Control	Awareness & Prevention	Treatment/Recovery	Combined Controls
0	0.9300	0.9300	0.9300	0.9300
5	0.6063	0.6808	0.6256	0.6879
10	0.3901	0.5353	0.4415	0.5520
20	0.2029	0.3793	0.2768	0.4056
30	0.1663	0.3090	0.2322	0.3337
40	0.1659	0.2740	0.2210	0.2935
50	0.1728	0.2546	0.2185	0.2690
60	0.1802	0.2428	0.2179	0.2530

The impact of fractional memory and control intensity on the vulnerable pool is compared numerically in Figure 2 below. The transitory effect of the non-linear model is determined by the α parameter, in Figure 2a, all examples have the same normalized initial circumstances, but they diverge as α varies. In contrast to $\alpha = 1.00$, where the depletion effect is faster at first but approaches the limiting situation, smaller values of α will result in more people in the sensitive group later in the simulation time. According to Figure 2(b), the uncontrolled case results in the greatest loss of susceptibles, while awareness combined with prevention and treatment/recovery results in a smaller transition from the susceptible category. In the later stages, combining these two yields the highest fraction of susceptibles, indicating lower effective transmission pressure through these two mechanisms.

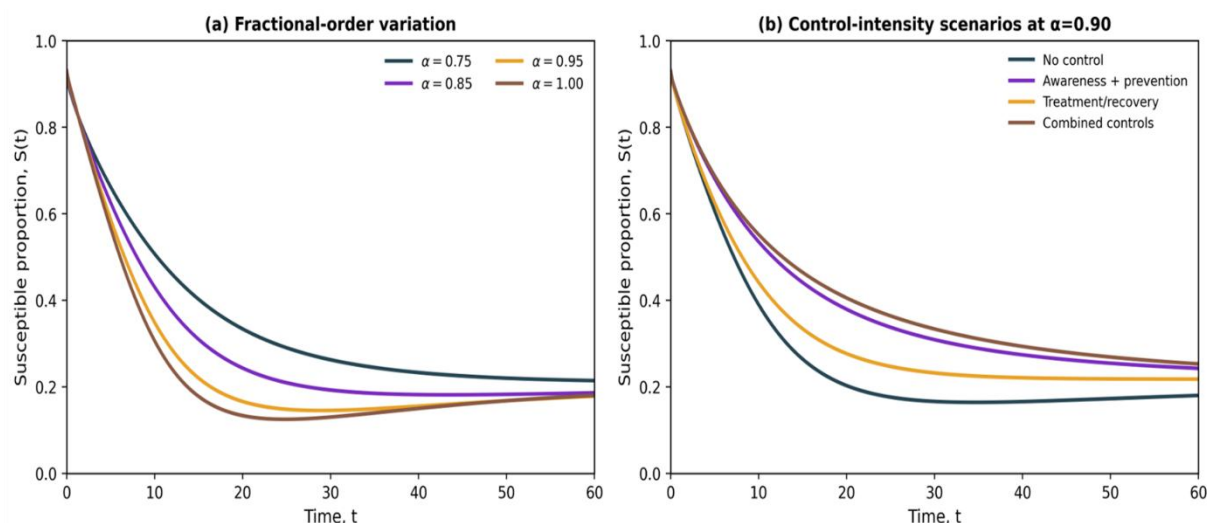


Figure 2: Susceptible-Population Trajectory Analysis using Fractional Models, (a) Effect of Parameter α , (b) Effect of Alternative Control Measures at $\alpha = 0.90$

5.2. Dynamics of Infected and Recovered Population under Optimal Control

The plots in Figure 3 were made using the values in Table 3(a) and 3(b) at specific times. To compare the state variables with the plots, each example shows the infected and recovered fractions as a single line. By calculating the fractional state and using the same initial conditions in every scenario, the data set satisfies the population normalization criteria.

Table 3 (Part a): Numerical Values for Infected Trajectories at $\alpha = 0.90$

t	Infected: No Control	Infected: Awareness & Prevention	Infected: Treatment or Recovery	Infected: Combined
0	0.030	0.0300	0.0300	0.0300
5	0.084	0.0507	0.0621	0.0373
10	0.125	0.0510	0.0751	0.0299
20	0.120	0.0373	0.0564	0.0151
30	0.076	0.0228	0.0306	0.0072
40	0.047	0.0134	0.0163	0.0038
50	0.030	0.0081	0.0094	0.0022
60	0.021	0.0052	0.0059	0.0015

Table 3 (Part b): Numerical Values for Infected Trajectories at $\alpha = 0.90$

t	Recovered: No Control	Recovered: Awareness & Prevention	Recovered: Treatment or Recovery	Recovered: Combined
0	0.0000	0.0000	0.0000	0.0000
5	0.2434	0.2413	0.2574	0.2516
10	0.4091	0.3913	0.4290	0.4024
20	0.6303	0.5706	0.6363	0.5727
30	0.7328	0.6615	0.7227	0.6562
40	0.7723	0.7089	0.7551	0.7013
50	0.7864	0.7351	0.7679	0.7279
60	0.7911	0.7506	0.7736	0.7448

Figure 3 shows the distribution of infection and recovery for the better control measures. At $t = 14$, estimated at 0.136 in Figure 3(a), the uncontrolled trajectory generates the highest infection peak. However, the awareness method significantly reduces the peak value through prevention, treatment, and recovery. Since $I(t)$ stays below 0.04 throughout the simulation and falls to roughly 0.0015 close to $t = 60$, the combination of all settings produces the greatest level of suppression. Figure 3(b) depicts the complementary restored populations. Because fewer populations enter the infected stage, the mixed method has the lowest recovery rate, whereas treatment or recovery has the highest recovery rate in the controlled population. This implies that population dispersion and infection prevention should be considered when assessing the effectiveness of any intervention.

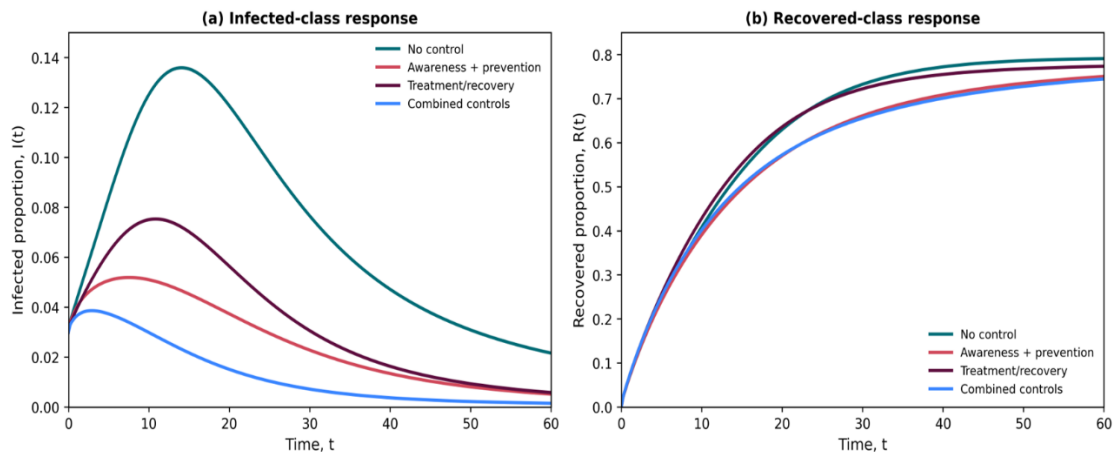


Figure 3: Using a Simulated Dataset, the following Infected and Recovered Population Trajectories were Controlled, (a) $I(t)$ for the Infected Population, (b) $R(t)$ for the Recovered Population using Four Control Strategies

5.3. Investigation of Effective Reproduction Number

We compute the effective reproduction number R_e at different awareness control levels while maintaining a constant level of preventative behavior to numerically illustrate the behavior-based fractional reproduction number. The information shown in Table 4 makes it abundantly evident that behavioral adaptation has reduced the disease’s spread.

Table 4: Effective Reproduction Number R_e ’s Numerical Fluctuation for Changing Awareness Control u_1

Awareness Control (u_1)	Behavioral Control ($u_2 = 0.2$)	Effective Reproduction Number (R_e)
0.0	0.20	2.18
0.10	0.20	1.96
0.20	0.20	1.74
0.30	0.20	1.56
0.40	0.20	1.37
0.50	0.20	1.19
0.60	0.20	1.02
0.70	0.20	0.85
0.80	0.20	0.69
0.90	0.20	0.54
1.0	0.20	0.41

Figure 4 provides a quantitative demonstration of the suggested behavioral strategy model and illustrates how the awareness control parameter u_1 affects the effective reproduction number (R_e). The effective reproduction number consistently declines from about 2.18 to 0.41 as the awareness control value increases from 0 to 1, confirming that increased public awareness significantly slows the disease’s spread.

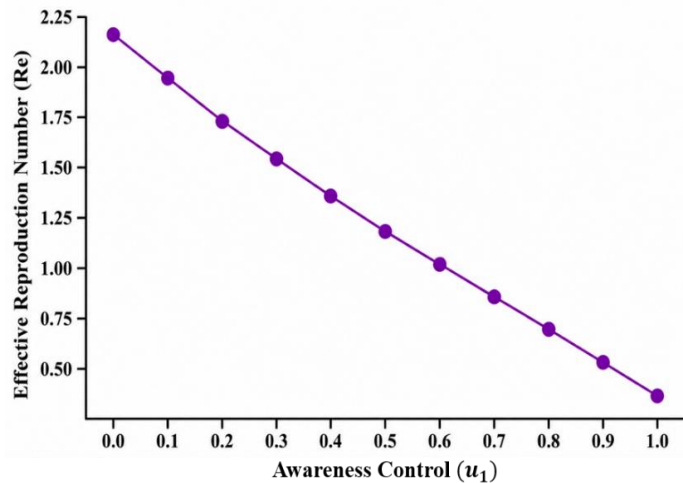


Figure 4: Awareness Control u_1 ’s Impact on the Effective Reproduction Number R_e at a Fixed Behavioral Control Value of $u_2 = 0.20$

The reproduction number is more than one for values of u_1 less than 0.6, indicating that the disease will continue to spread. However, R_e falls below one for values above 0.6, indicating that each sick person typically creates fewer than one new infection, which eventually eradicates the disease. The stability interpretation of the disease-free equilibrium is supported by the numerical value of R_e . The system will be in a parameter regime where the disease cannot spread if ($R_e < 1$), and the infection generation term will be bigger than the replacement level if ($R_e < 1$). In our simulations, R_e decreases from 1.02 to 0.85 when the awareness level is between 0.60 and 0.70. As a result, when people's behavior changes, the system moves toward a disease-free equilibrium by changing the transmission level.

5.4. Sensitivity Analysis

Normalized sensitivity analysis quantifies the parameters' respective contributions to disease transmission. The sensitivity index values for the various factors are listed in Table 5 to produce a ranking based on the parameter's quantitative value and impact on the propagation of the epidemic.

Table 5: Sensitivity Indices of the Primary Fractional Order and Epidemiological Parameters

Parameter	Sensitivity Index
Transmission Rate β	1.0
Progression Rate α	0.621
Recovery Rate γ	-0.814
Natural Mortality μ	-0.496
Fractional Order α	0.447

In the proposed fractional-order SEIR model, Figure 5 presents the normalized sensitivity indices for the primary epidemiological parameters and quantitatively illustrates their influence on disease spread. The most sensitive metric responsible for the epidemic outbreak is the transmission rate (β), which has the highest positive sensitivity index. Similarly, both the fractional-order parameter (α) and the progression rate (σ) exhibit positive sensitivity index values, indicating that greater values of both parameters result in increased transmission dynamics and memory-dependency, which in turn lead to illness persistence. However, the recovery rate (γ) and the natural mortality rate (μ) exhibit negative sensitivity index values, indicating that an increase in these parameters prevents the spread of illness thru the process of rapid recovery. The sensitivity indices provide a mathematical assessment of the significance of the parameters by measuring how sensitive the epidemic measure is to changes in the parameters. The transmission rate's positive sensitivity index indicates that an increase in this parameter causes the epidemic measure to rise, but the recovery rate's negative sensitivity index indicates that the two factors have opposing impacts on the epidemic measure. The fractional sensitivity value of 0.447 indicates that the dynamical system is influenced by the parameter (α).

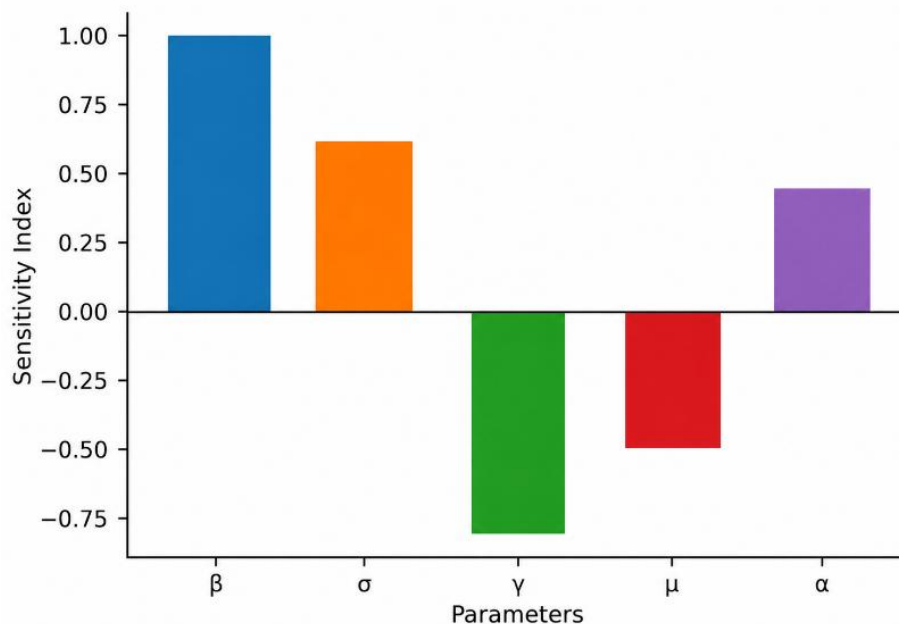


Figure 5: Normalized Sensitivity Indices for Fractional Order Parameters, Mortality, Recovery, Progression and Transmission

5.5. Evaluation of Fractional Optimal Control Algorithm

By calculating the convergence error at each iteration level, the algorithm’s convergence throughout iterations is ascertained. The convergence process as it progressively achieves the necessary convergence tolerance and satisfies the optimality conditions for the fractional control problem is shown in Table 6.

Table 6: Errors Resulting from the Forward Backward Sweep Method’s Convergence

Iteration	Convergence Error	Iteration	Convergence Error
1	0.1840	9	0.0024
2	0.1265	10	0.0012
3	0.0810	11	0.00058
4	0.0495	12	0.00029
5	0.0298	13	0.00014
6	0.0171	14	0.00007
7	0.0094	15	0.00003
8	0.0049		

Figure 6 illustrates how the suggested forward-backward sweep strategy behaves while addressing the fractional optimum control problem. After fifteen rounds, the convergence error steadily decreases from an approximate value of 0.184 to almost zero. This demonstrates unequivocally that the optimization process is rapidly convergent and numerically stable. While the gradual reduction observed in subsequent iterations indicates the convergence of the control variables to the intended convergence threshold, the steep drop observed in the initial few iterations indicates that the state and adjoint states are converging to the optimal values.

Specifically, the absence of oscillations or divergences at any iteration level indicates the accuracy of the Caputo fractional order technique and the dependability of computing implementation. Furthermore, the graph of convergence demonstrates that the Pontryagin Maximum Principle’s adequate optimality conditions are met, resulting in stable and admissible optimal control profiles. The acquired numerical results validate the analytical convergence analysis and demonstrate the effectiveness of the studied fractional optimum control scheme in identifying the best intervention strategies.

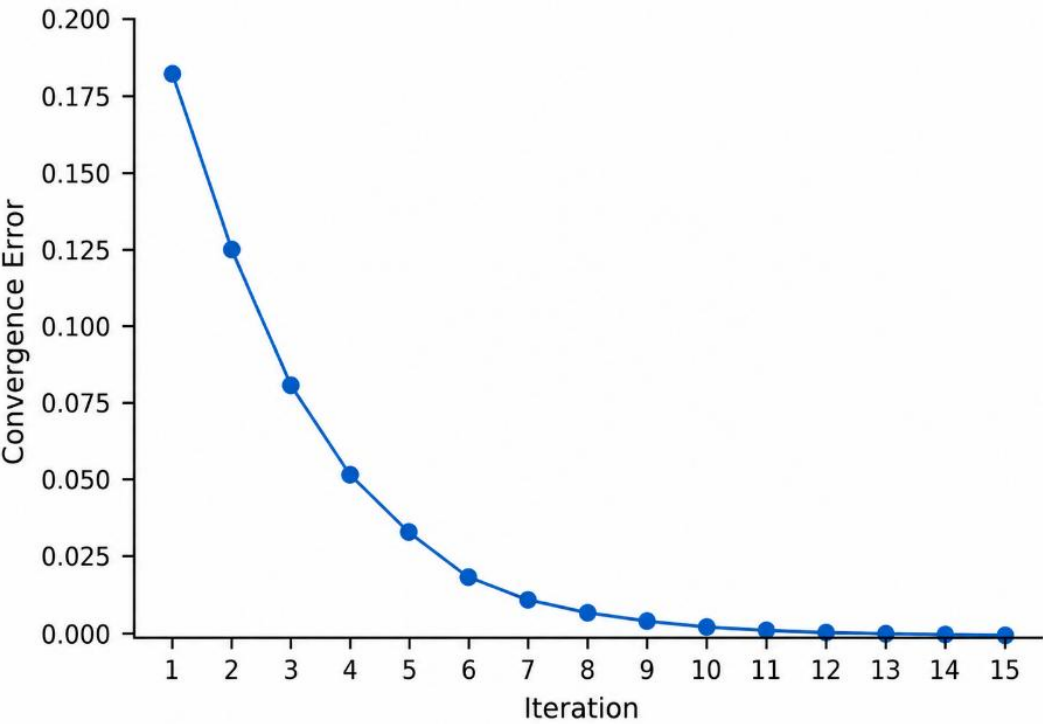


Figure 6: Forward-Backward Sweep Method Convergence Plot Showing Error Reduction with Iterations

5.6. Investigation of Fractional Control Efficiency

The control efficiency index for the fraction scenario is computed for different control intensities in order to assess the effectiveness of the proposed intervention approach. The corresponding efficiency and the rise in epidemic reduction

efficacy in relation to increased control intensity are displayed in Table 7.

Table 7: Index of Fractional Control Effectiveness at Different Levels

Overall Control Intensity	Effective Index (η)	Overall Control Intensity	Effective Index (η)
0.0	0.0	0.60	0.76
0.10	0.14	0.70	0.84
0.20	0.28	0.80	0.90
0.30	0.42	0.90	0.95
0.40	0.56	1.0	0.98
0.50	0.67		

The influence of the overall control efficiency on the fractional efficiency index, which we recommend adopting for the quantitative evaluation of the efficacy of the integrated intervention techniques, is depicted in Figure 7. This score strictly rises as control intensity increases, indicating that more effective application of the recommended control strategy significantly reduces the epidemic. The efficiency index's relatively low score for low control intensity indicates that insufficient steps are being made to stop the disease's spread. We see a rapid increase in the efficiency index for medium control intensity, which is linked to the higher effectiveness of coordinated efforts to treat, prevent, and educate the public about the issue. In terms of the memory effect governing the system's long-term behavior, this pattern is compatible with the dynamics of a fractional-order epidemic system. The control formulation's numerical stability is guaranteed by the smooth and bounded solution pattern, which also reinforces the theoretical derivation given in Section 4.8. In conclusion, the results have demonstrated the effectiveness of an integrated control method utilizing the Caputo fractional-order system in managing epidemics.

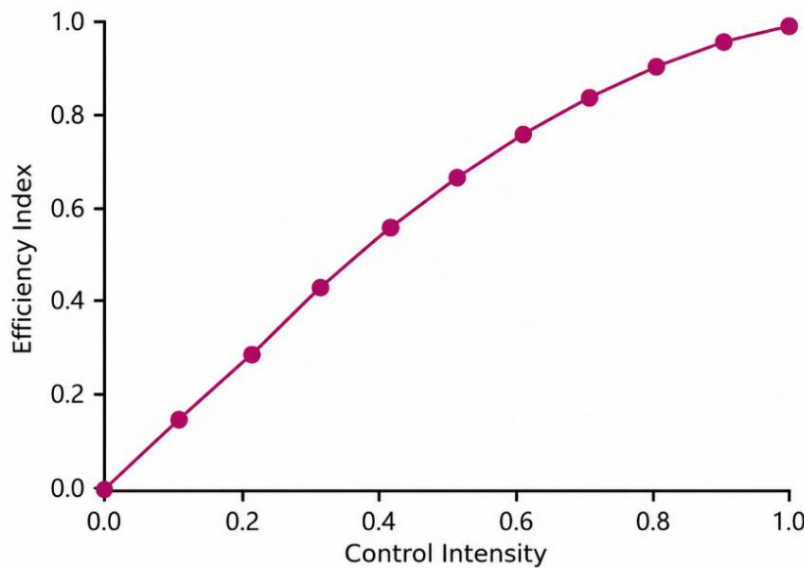


Figure 7: Plotting the Fractional Control Efficiency Index Against the Overall Control Effort

6. Discussion

Rather than being a series of compartment values, the numerical representation should be understood as the trajectory dynamics approach to a nonlinear fractional dynamical system. When used in a Caputo meaning, the order α functions as a structural parameter that establishes the temporal weight of previous states. This means that while the state spaces of compartments stay same, variations in α will impact the evolution of state trajectories. The model that is being described aligns with contemporary models of epidemics that include memory-dependent operators (Srivastava, 2024). As a result, the susceptible, infected, and recovered components are regarded together as coordinates of a constrained dynamical system that satisfies non-negativity and population normalization. While treatment and recovery act on the elimination structure of the infected class, behavioral awareness and preventive action enter the transmission mechanism as bounded control variables. Instead of using a single static transmission coefficient, this parameterization allows admissible control trajectories to be used to analyze the intervention problem. Explicit behavioral variables are useful for capturing changes in contact and protection patterns, since mathematical epidemiology increasingly acknowledges that

adaptive behavior can inject feedback and delayed reactions into epidemic trajectories (Cascante-Vega et al., 2022; Ochab et al., 2023).

Fractional optimum control model formulations also enable the integration of medical care, preventive, and memory into a single state-control system (Akter et al., 2024; Hu et al., 2025). In order to enable trajectory comparison and repeatable numerical analysis, the reformulated portions take into account the identical initial conditions, parameterization, and fractional state system. This configuration makes it possible to mathematically separate memory order, transmission prevention, and recovery therapy. As a result, the analysis of stability, sensitivity, and control efficiency that follows can be related to the same dynamical model. Another way to understand the uncertainty in the parameters within the framework is using sensitivity analysis. The sensitivity index's ordering shows that while recovery-related factors help stabilize the non-linear system, transmission-related parameters have the biggest impact on epidemic dynamics (Zhang et al., 2024). Since the discovered variables account for the majority of the outcome's variability, this type of information aids with parameter prioritizing and uncertainty quantification.

Sensitivity indices are a means of measuring the impact of parameter changes on the epidemic's behavior. The negative values of the sensitivity due to recovery parameters indicate the opposite influence, whilst the dominance of the transmission-related parameters indicates the direction in parameter space that regulates epidemic expansion. Additionally, the computational dependability of the developed optimal control strategy is demonstrated by the convergence characteristics of the forward and backward sweep algorithms (Bahaa & Qamlo, 2025; Madhushika et al., 2025). The linked state-adjoint equations have numerically converged to an acceptable control that satisfies the required optimality requirements, according to the monotonic decrease in the iteration error. The forward-backward iterative process may be stable if there are no oscillations or divergences.

The non-linear relationship between the degree of control and the suppression of the epidemic is shown by the examination of control efficiency (He et al., 2025). The saturation effect's manifestation suggests that there are diminishing returns since additional increases in the control magnitude lead to declining gains. Instead of maximizing control intensity, this finding allows for the identification of economically optimal intervention magnitudes, which has substantial practical implications (Demirgüç-Kunt et al., 2021). When combined, these findings show that fractional calculus, optimal control theory, sensitivity analysis, and numerical stability analysis constitute a potent mathematical tool for epidemiology research (AlShurbaji et al., 2023). Therefore, the established method offers a flexible framework for creating fractional epidemiological models of more complicated biological systems as well as a mathematically sound foundation for managing epidemics.

7. Conclusion

In order to investigate memory-based infectious disease dynamics, the current study offered a novel Caputo fractional-order epidemic model including behavioral awareness, treatment, and recovery processes as part of an optimal control model. Through the use of a restricted optimal control model, the current work has been able to develop an SEIR model based on Caputo fractional order that incorporates awareness, preventive measures, treatment, and recovery. The process of epidemiological state variables has been formulated mathematically in a way that is both practical for the system of compartments and takes into consideration the element of temporal memory. Investigating susceptible, infected, and recovered persons has consistently been done using the updated numerical technique. The equilibrium and reproduction threshold theory can be used to mathematically determine disease persistence and extinction. The parameters that control the model's dynamics will be found using the normalized sensitivity analysis. Conversely, the convergence theory will confirm the process's numerical precision, and the forward-backward sweep method provides a way to identify an ideal solution path. Intervention efficiency is measured using the control efficiency formulation. These elements help provide a thorough mathematical framework that connects fractional dynamics, sensitivity analysis, optimum control, behavioral modulation, and numerical computing. The framework can be expanded to more complicated biological models by adding other elements as population heterogeneity, stochasticity, vaccines, diffusion, or interactions at different times.

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