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Quantum Mechanics for Population Dynamics: A New Approach to Population Dynamics, A Study of Immigration, Emigration and Fission via Quantum Mechanics

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ABSTRACT

A standard single species immigration, emigration and fission ordinary differential equation (ODE) model is derived from a quantum mechanics approach. The stochasticity introduced via quantum mechanics is very different than that of the standard approaches such as demographic stochasticity in the state variables or environmental stochasticity as in uncertainty quantification. This approach yields a standard ODE and predicts the effects of quantum tunneling of probabilities. This approach is explained in such a way that epidemiologists, mathematicians, mathematical biologists, etc who are not familiar with quantum mechanics can understand the methods described here and apply them to more sophisticated situations. The two main results of this approach are (i) standard macroscopic ODE models can be derived from first principles of quantum mechanics instead of making macroscopic heuristic assumptions and (ii) high impact events with low probability of occurrence can be explicitly calculated.

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1 Introduction

To help provide informed recommendations to public health policy makers, researchers often construct standard models of population dynamics using ordinary differential equations (ODE) or partial differential equations (PDE). These macroscopic models make heuristic assumptions such as competition, decay, growth, homogeneous mixing, etc. and yield models that can be thoroughly analyzed. Features such as equilibrium points, stability and what-if questions such as sensitivity analysis or uncertainty quantification can be answered (Chowel et al., 2009, pp. 195–247).

When the deterministic results are compared with real data, there are always discrepancies. In order to make the ODE/PDE models better agree, noise is introduced via stochasticity. Usually this is done in one of two ways: introduce demographic stochasticity via a Wiener process where the randomness is incorporated into the state variables of the model, and/or account for environmental stochasticity as randomness introduced via the parameters of the model-uncertainty quantification. Demographic and environmental stochasticity does provide a degree of controlled randomness, however at the expense of obtaining analytic results. Usually, numerical simulations are the only means by which to analyze the model. From a practical sense, large numbers of runs are needed in order to get a sense of how the model reacts to fluctuations in the state variables or the parameters. Uncertainty quantification does give some useful information but doesn't give precise predictions. This paper utilizes the quantum machinery developed in (Baez and Biamonte, 2018) and applies it to an immigration, emigration and fission Petri net model (Koch et al., 2011). An appropriate Schrodinger equation (SE) PDE will be constructed. The quantum expected value operator acting on the solution of the SE reduces to a classical standard ODE. This new result gives a solid foundation for a justification of macroscopic ODEs without the usual heuristic assumptions. This surprising result suggests that other commonly used ODE models might have a solid justification based on quantum mechanics rather than heuristic assumptions.

The quantum mechanics approach developed and analyzed here introduces stochasticity via the probability of specific interactions between individual objects instead of demographic or environmental stochasticity. Interactions between individual

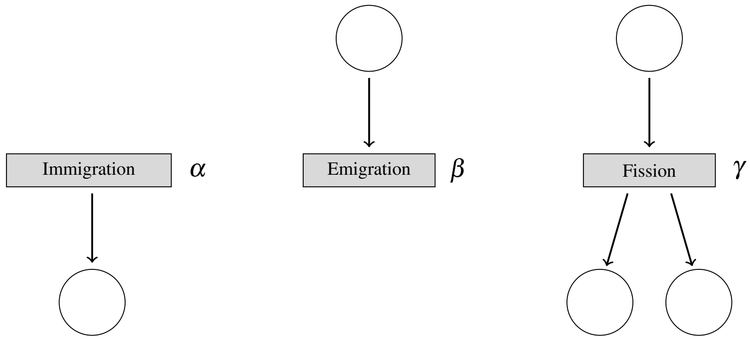

Figure 1: Immigration rate α, emigration rate β, fission rate γ.

objects occur at the quantum level. These interactions can increase, decrease or not change the single species population. Specific results are derived as to how the probabilities of these interactions change as a function of time.

The key to providing the necessary and appropriate mathematical machinery (Baez and Biamonte, 2018) will be discussed in a later section. A combinatorial argument will provide a consistent way of counting the number of ways objects can be created or destroyed. This approach provides a formal way of calculating the changes in the probability densities. Specifically, $\phi_0(t)$ is the probability of having exactly 0 objects at any time $t \geq 0$. Similarly, $\phi_2(t)$ and $\phi_{11}(t)$ are the probabilities of having exactly 2 or 11 objects at time t respectively. These discrete probability densities $\phi_n(t)$ provide the vehicle for the quantum tunneling of probabilities whereby unlikely events can occur, albeit with small probability.

The temporal evolution of the system will be determined via the differentiable Markov generating function (Wilf, 1990) (GF) Φ where

$$\Phi(t,z) = \sum_{n=0}^{\infty} \phi_n(t) z^n.$$

Generating functions act as a conduit between discrete and continuous mathematics.

> "A generating function is a clothesline on which we hang up a sequence of numbers for display." (Wilf, 1990)

The clothespins are the monomials z^n and the individual laundry items are the probabilities $\phi_n(t)$. A GF is written as an infinite power series where the coefficients of the monomials are the objects of interest. The monomials here act as place holders and do not have any physical role in the analysis. In this paper we will not discuss other generating functions such as Dirichlet, exponential, etc. generating functions. This GF describes the temporal probability $\phi_n(t)$ of having exactly n objects at any time t. Additionally, the analytic properties of the formal series $\Phi(t,z)$ will not be considered. The reason for ignoring whether the series is/is not convergent is that the manipulations that will be performed are defined over the product topological ring of formal power series.

2 Mathematical Methods

2.1 Immigration, emigration and fission model

Consider a model where we assume that the only important population dynamical interactions are immigration, emigration and birth via fission (Brauer and Castillo-Chavez, 2012; Edelstein-Keshet, 1988; Maki and Thompson, 1973). Assume that the immigration rate is constant (α), the emigration rate is proportional to the current population (β), and the rate of fission is proportional to the current population (γ). The associated continuous Markov chain is shown in Figure 1.
From the classical approach the standard ODE model is given by

$$\mu'(t) = (\gamma - \beta)\mu + \alpha \tag{1}$$

with initial condition (IC) $\mu(t=0) = \mu_0$. The equilibrium point $\bar{\mu} = \alpha/(\beta - \gamma)$ is non-negative and stable provided $\beta > \gamma$.

The two main results of this paper are:

i This macroscopic ODE model (1) will be derived using quantum mechanics and

ii the probability densities $\phi_n(t)$ will suggest that huge numbers of objects are possible, albeit with small probability.

Now consider the above model but from a quantum approach. Notice that there are 3 components to each individual Petri net (Koch et al., 2011). The state that the system is in before an interaction occurs is depicted above the rectangle. The rectangle denotes the type of interaction that can occur. Lastly, the state of the resulting interaction is denoted as below the rectangle. It should be noted that all the interactions can take place simultaneously.

Consider the mechanism of immigration where an object(s) move from some other locale to now being part of the system. We could alternatively describe this transition as pure birth at constant rate α. The second mechanism of emigration is where an object(s) leaves the current state for some other region at the rate of proportion β to the current population. This could also be considered as pure death. The third mechanism is that of fission or asexual reproduction at the rate of proportion γ to the current population.

The quantum approach developed later yields the Schrodinger equation

$$\frac{\partial \Phi}{\partial t} + \left(-\gamma z^2 + (\beta + \gamma)z - \beta\right) \frac{\partial \Phi}{\partial z} = \alpha\,(z-1)\,\Phi \tag{2}$$

with initial condition (IC)

$$\Phi(t=0, z) = z^{\mu_0},$$

where $\mu_0 \in \mathbb{N}$ and denotes the initial number of objects present at time $t = 0$. The boundary condition (BC) is

$$\Phi(t, z=1) = 1$$

and reflects the fact that the sum of the individual probabilities must be 1. That is $\sum_{n=0}^{\infty} \phi_n(t) = 1$. Using this condition we find

$$\left.\frac{\partial \Phi(t,z)}{\partial t}\right|_{z=1} = 0, \qquad \text{and} \qquad \left.\Phi(t,z)\right|_{t=0} = 1 \cdot z^{\mu_0}.$$

We will discuss the derivation of this Schrodinger equation as well as the IC and BC in the next section. Solving the Schrodinger equation yields the closed form solution

$$\Phi(t,z) = \left(\beta-\gamma\right)^{-\frac{\alpha}{\gamma}} \left[\frac{\beta(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)t}}{\gamma(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)t}}\right]^{\mu_0} \left[\frac{(\beta-\gamma)^2 e^{(\beta-\gamma)t}}{\gamma(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)t}}\right]^{\alpha/\gamma}. \tag{3}$$

The specific functions/operators that are of interest are the probability densities $\phi_n(t)$ and a quantum expected value operator. This operator $\mathbb{E}$ defined later acts on the GF Φ and yields the time dependent function μ

$$\mathbb{E} \circ \Phi = \mathbb{E}[\Phi]\big|_{z=1} =: \mu(t).$$

The quantum approach describes the interactions at the individual particle (object level) and yields the surprising standard macroscopic ODE model as given in (1). This suggests that there is an intimate connection between the quantum and macroscopic scale dynamics. This result also provides a justification and explanation for the large scale heuristic ODE model. In other words, the expected value of the quantum probability density fluctuations provides the specific details of why a heuristic model such as (1) can be thought of as a cartoon of these processes.

2.2 Generating functions of the random variable

We now discuss the implications of the quantum paradigm that cannot be deduced from the macroscopic viewpoint as given by the standard single species population model given in equation (1).

By expanding the explicit GF given in equation (3) the first two densities $\phi_0(t)$ and $\phi_1(t)$ are given by

$$\phi_0(t) := \left(\frac{(\beta-\gamma)e^{\beta t}}{\beta e^{\beta t} - \gamma e^{\gamma t}}\right)^{\alpha/\gamma} \left(\frac{\beta\left(e^{\beta t} - e^{\gamma t}\right)}{\beta e^{\beta t} - \gamma e^{\gamma t}}\right)^{\mu_0}$$

and

$$\phi_1(t) := \frac{\left(\frac{(\beta-\gamma)e^{\beta t}}{\beta e^{\beta t}-\gamma e^{\gamma t}}\right)^{\alpha/\gamma} \left(\frac{\beta\left(e^{\beta t}-e^{\gamma t}\right)}{\beta e^{\beta t}-\gamma e^{\gamma t}}\right)^{\mu_0-1} \left(\alpha\beta e^{2\gamma t} + \alpha\beta e^{2\beta t} + e^{t(\beta+\gamma)}\left(\mu_0(\beta-\gamma)^2 - 2\alpha\beta\right)\right)}{\left(\beta e^{\beta t} - \gamma e^{\gamma t}\right)^2}.$$

Obviously these discrete density functions are extremely complicated. In order to illustrate quantum tunneling of probabilities, we examine a special case. Strictly speaking, the quantum tunneling effect is where there is some kind of a barrier whereby in

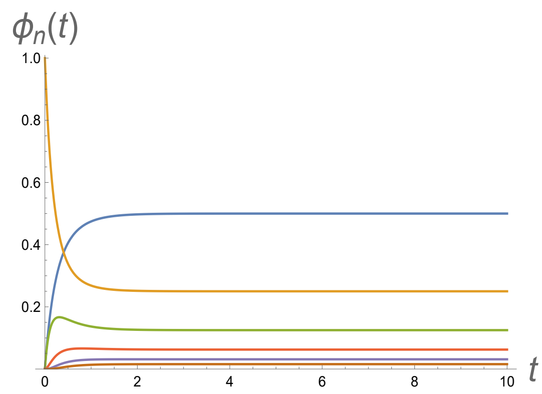

Figure 2: Truncated probablity densities $\phi_0, \ldots, \phi_5$ vs. time t.

classical mechanics a particle cannot penetrate. In quantum mechanics there does exists the possibility that the particle will penetrate and pass through the barrier.

The above standard ODE model predicts that a positive and stable equilibrium point is given by

$$\bar{\mu} := \frac{\alpha}{\beta - \gamma},$$

provided $\gamma < \beta$. If we choose $\alpha = \gamma = 1, \beta = 2$, and $\mu_0 = 1$ then the equilibrium point $\bar{\mu} = 1$ is stable and the GF reduces to the much simpler expression

$$\Phi(t, z) = \frac{e^t \left(e^t (2 - z) + 2(z - 1)\right)}{\left(e^t (2 - z) + z - 1\right)^2}. \tag{4}$$

The individual densities reduce to

$$\begin{aligned}
\phi_0(t) &= \frac{2e^t \left(e^t - 1\right)}{\left(2e^t - 1\right)^2} \\
\phi_1(t) &= \frac{e^t \left(2e^{2t} - 3e^t + 2\right)}{\left(2e^t - 1\right)^3} \\
\phi_2(t) &= \frac{2e^t \left(e^t - 1\right) \left(e^{2t} - e^t + 1\right)}{\left(2e^t - 1\right)^4} \\
\phi_3(t) &= \frac{e^t \left(e^t - 1\right)^2 \left(2e^{2t} - e^t + 2\right)}{\left(2e^t - 1\right)^5} \\
\phi_4(t) &= \frac{2e^t \left(e^t - 1\right)^3 \left(e^{2t} + 1\right)}{\left(2e^t - 1\right)^6} \\
\phi_5(t) &= \frac{e^t \left(e^t - 1\right)^4 \left(2e^{2t} + e^t + 2\right)}{\left(2e^t - 1\right)^7} \\
&\vdots
\end{aligned}$$

The graphs of the first 6 densities are shown in Figure (2). Note that the probabilities (as $t \longrightarrow \infty$) satisfy the decreasing monotonicity condition $\phi_{n+1} \leq \phi_n$. It can be shown that the stationary solutions satisfies the condition $\lim_{t \to \infty} \phi_n(t) = (1/2)^{n+1}$ for $n \in \mathbb{N}$. Now compare the prediction made by the macroscopic model versus the quantum model. Specifically, if we start with one object, the macroscopic ODE model predicts that as $t \longrightarrow \infty$ the stable equilibrium point is $\bar{\mu} = 1$. The quantum model however predicts that as $t \longrightarrow \infty$ the probability of having zero objects is $\frac{1}{2}$, the probability of having one object is $\frac{1}{4}$, etc. By extension, the probability of having 15 objects is $\frac{1}{2^{16}}$, small but not zero. The quantum model given in equation (3) predicts a quantum tunneling effect of probabilities as a type of noise that is not captured by the standard deterministic ODE model given in equation (1).

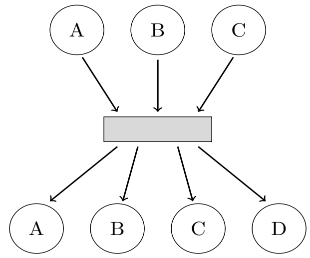

Figure 3: $\{A, B, C\} \to \{A, B, C, D\}$.

3 A Quantum Analogy

This section contains an abbreviated discussion of quantum/stochastic concepts (Baez and Biamonte, 2018) that are applicable to this work.

3.1 Creation and annihilation operators

In quantum mechanics the lead actors in the play are the creation operator a^+ and the annihilation operator a^- (Note: Physicists denote the annihilation operator as simply a). The operator a^+ takes an object from energy level $\mathcal{E}_n$ and moves it up one level to $\mathcal{E}_{n+1}$. The operator a^- takes an object from energy level $\mathcal{E}_n$ and moves it down one level to $\mathcal{E}_{n-1}$. Analogously, in population dynamics we define the creation operator a^+ to take n objects and turn them into $n+1$ objects. The annihilation operator a^- destroys one of the n objects into $n-1$ objects.

3.1.1 Growth and the creation operator a^+

Consider the scenario where we initially have three objects A, B and C, and then add one generic external object D as seen in Figure 3. Notice there is only one way of adding an additional generic object D. Now introduce a monomial z^3 which represents the fact that initially there are exactly three objects. Please note that the variable z is independent of time t and has no physical meaning. The only point of interest here is the exponent 3 of the monomial. In other words, we can think of the monomial z^3 as a placeholder. After the interaction has occurred, there are now exactly four objects, which we associate with the monomial z^4. This means that the initial monomial z^3 now becomes z^4. We assume that this holds true for any initial number of n objects, in which case by induction the initial monomial z^n becomes z^{n+1}, $\forall n \in \mathbb{N}$. The creation operator a^+ simply multiplies the monomial z^n by z yielding z^{n+1}. Hence we define the action of the creation operator a^+ to be

$$a^+ [z^n] := z^{n+1}, \qquad \forall n \in \mathbb{N}. \tag{5}$$

3.1.2 Decay and the annihilation operator a^-

Now consider the reverse scenario where three objects interact, but now one object is annihilated, resulting in two remaining objects. Since there are three ways for two objects to remain in existence, as seen in Figure 4, we associate the action of a^- to be $a^-[z^3] = 3z^2$. We assume this holds true for any $n \in \mathbb{N}$ objects, the initial monomial z^n becomes nz^{n-1}, in which case the annihilation operator a^- is simply the usual derivative operator $\partial/\partial z$. Hence, we define the action of the annihilation operator a^- to be

$$a^- := \frac{\partial}{\partial z}, \tag{6}$$

and by induction define

$$(a^-)^n := \frac{\partial^n}{\partial z^n}, \qquad \forall n \in \mathbb{N}. \tag{7}$$

3.1.3 Combinatorial meaning of the operators a^+ and a^-

Consider the interaction where two objects interact sexually and produce a single offspring with rate λ as shown in Figure 5. In order to motivate how the number of ways that objects change from the input side $j = 2$ as compared to the output side $k = 3$ we discuss the formulation of the Hamiltonian operator as found in (Baez and Biamonte, 2018). A Hamiltonian operator can be thought of in at least two different ways. The first is a change in energy as in the quantum approach; the second is the change in the number of ways that the output side changes minus the number of ways the input side does not change as in the

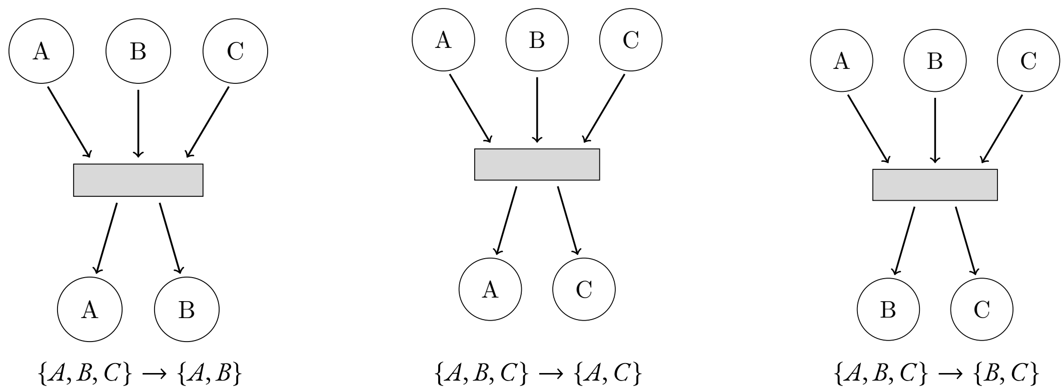

Figure 4: Eliminate one object from $\{A, B, C\}$.

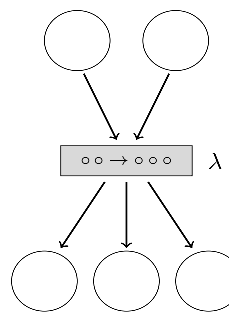

Figure 5: Sexual reproduction with rate λ.

quantum/stochastic approach here. To simplify the idea to its most basic form, we informally define the Hamiltonian operator as

$$\mathcal{H} := \text{Final State} - \text{Initial State}.$$

The expressions Final State and Initial State need to be appropriately defined.

For this example, consider the action of annihilating two objects on the input side, followed by the action of creating two objects. In other words, the total number of ways that two objects do not change via the creation and annihilation operators is given by

$$\left(a^{+}\right)^{2}\left(a^{-}\right)^{2} = z^{2}\frac{\partial^{2}}{\partial z^{2}}.$$

In order to understand the combinatorial interpretation, consider the action of $(a^{+})^{2}(a^{-})^{2}$ on an arbitrary monomial such as z^{5}, that is

$$\left(a^{+}\right)^{2}\left(a^{-}\right)^{2}\left[z^{5}\right] = 5 \cdot 4 \cdot z^{5}.$$

The combinatorial interpretation is: How many ways can we annihilate 2 objects out of 5 objects ($\partial^{2}/\partial z^{2}$) and then followed by bringing back 2 objects (z^{2}). In other words, this action is nothing more than the permutation $P(5, 2)$. Moreover, this means $P(5, 2)$ is the total number of ways that nothing has changed, hence this is how we calculate the number of ways the initial state does not change.

Next, consider the action of $(a^{+})^{3}(a^{-})^{2}$ on an arbitrary monomial such as z^{7}, that is

$$\left(a^{+}\right)^{3}\left(a^{-}\right)^{2}\left[z^{7}\right] = 7 \cdot 6 \cdot z^{8}.$$

The combinatorial interpretation is: How many ways can we annihilate 2 objects out of 7 objects ($\partial^{2}/\partial z^{2}$) and then followed by bringing back 3 objects (z^{3}). In other words, this action is how we calculate the final state. The net change is defined as the Hamiltonian operator

$$\mathcal{H} := \left(a^{+}\right)^{3}\left(a^{-}\right)^{2} - \left(a^{+}\right)^{2}\left(a^{-}\right)^{2}.$$

In general, the scenario where j distinct objects enter into an interaction and k objects emerge is shown in Figure 6. We

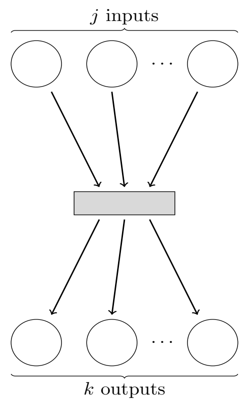

Figure 6: j objects enter into an enteraction & k objects emerge.

describe these type of processes where the net change is quantified by the change in the number of the objects via the Hamiltonian operator

$$\mathcal{H}\,[\overbrace{\circ\cdots\circ}^{j\text{-inputs}} \to \overbrace{\circ\cdots\circ}^{k\text{-outputs}}].$$

Towards this goal, the linear operator $\mathcal{H}$ will be a modified Hamiltonian operator from quantum mechanics and will be composed of suitably modified creation and annihilation operators as has been defined in (Baez and Biamonte, 2018).

First, consider the input side of the interactions. We quantify the scenario where all distinct input objects are annihilated followed by the action where all destroyed objects are then recreated. In other words, we are counting the total number of ways that the input configuration is unchanged. In general, if there are j objects in the initial configuration then the action is given by

$$(a^+)^j\,(a^-)^j = z^j\,\frac{z^j}{\partial z^j}.$$

Next, examine the output configuration which is defined by annihilating j inputs and then recreating k outputs. The action of this process is given by

$$(a^+)^k\,(a^-)^j = z^k\,\frac{z^j}{\partial z^j}.$$

The stochastic Hamiltonian as defined in (Baez and Biamonte, 2018) is the difference between the final and initial configurations and is given by the stochastic Hamiltonian for the homogeneous class of j-inputs and k-outputs as follows:

$$\mathcal{H}_{j,k} := \lambda\Big[\overbrace{(a^+)^k}^{\text{Create } k \text{ outputs}} \circ \underbrace{(a^-)^j}_{\text{Annihilate } j \text{ inputs}} - \overbrace{(a^+)^j}^{\text{Create } j \text{ inputs}} \circ \underbrace{(a^-)^j}_{\text{Annihilate } j \text{ inputs}}\Big], \tag{8}$$

where λ is the rate at which the interaction occurs.

As an example, consider the sexual reproduction interaction as shown in Figure 5. The associated Hamiltonian is

$$\mathcal{H} := \lambda\left(z^3 - z^2\right)\frac{\partial^2}{\partial z^2},$$

in which case the Schrodinger equation is

$$\frac{\partial\Phi}{\partial t} = \lambda\left(z^3 - z^2\right)\frac{\partial^2\Phi}{\partial z^2}$$

and describes the temporal evolution of the GF Φ.

Since the derivative is a linear operator then the Hamiltonian operator is also linear. Hence each individual interaction in the immigration, emigration and fission Petri nets can be constructed and the entire model can be written as the sum of the interactions.

3.2 Expected value operator of the generating function

Suppose we are interested in the expected number of objects at time t. Consider the number operator defined as

$$\begin{aligned} N &:= a^{+}a^{-} \\ &= z\frac{\partial}{\partial z}. \end{aligned}$$

The reason it is called the number operator is that its action basically returns the number of objects as seen here

$$\begin{aligned} N\left[z^{k}\right] &= z\frac{\partial}{\partial z}\left[z^{k}\right] \\ &= kz^{k} \\ (N - k\mathbf{1})\, z^{k} &= \mathbb{O}, \end{aligned}$$

in which case

$$N = k\mathbf{1}.$$

Now consider the action of the number operator N on the GF

$$N\left[\sum_{n=0}^{\infty}\phi_n(t)z^n\right] = \sum_{n=0}^{\infty} n\phi_n(t)z^n.$$

Next, define the expected value operator as

$$\mathbb{E}[\cdot] := z\frac{\partial}{\partial z}[\cdot]\bigg|_{z=1}$$

and lastly define the first moment $\mu(t)$ as

$$\mu(t) := \mathbb{E}\left[\Phi(t;z)\right] = N \circ \Phi(t;z)\bigg|_{z=1} = \sum_{n=0}^{\infty} n\phi_n(t). \tag{9}$$

The standard macroscopic ODEs describing population dynamics will be derived and be of the form

$$\frac{d\mu}{dt} = f(\mu).$$

This quantum approach will surprisingly yield standard models such as the linear equation given in (1).

3.2.1 An alternative way of calculating the expected value

The SE can be written in operator form

$$\frac{\partial\Phi}{\partial t} = \mathcal{H} \circ \Phi, \tag{10}$$

where $\mathcal{H}$ is the associated Hamiltonian operator. Since the variables t and z are independent, consider the commutator operator

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial z}\right] := \frac{\partial}{\partial t}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\frac{\partial}{\partial t} = \mathbb{O}.$$

Premultiply (10) by the number operator $\mathcal{N} = z\frac{\partial}{\partial z}$ and using the commutative property we find the alternative way of calculating the mean as follows

$$\begin{aligned} \mathcal{N}\frac{\partial\Phi}{\partial t} &= \mathcal{N}\left[\mathcal{H} \circ \Phi\right] \\ z\frac{\partial}{\partial z}\frac{\partial\Phi}{\partial t} &= z\frac{\partial}{\partial z}\left[\mathcal{H} \circ \Phi\right] \\ \frac{\partial}{\partial t}\left[z\frac{\partial\Phi}{\partial z}\right]\bigg|_{z=1} &= z\frac{\partial}{\partial z}\left[\mathcal{H} \circ \Phi\right] \\ \frac{d\mu}{dt} &= \frac{\partial}{\partial z}\left[\mathcal{H} \circ \Phi\right]\bigg|_{z=1}. \end{aligned} \tag{11}$$

4 Immigration, Emigration and Fission Model

Consider a single species population undergoing the parallel processes of immigration, emigration, and fission as shown in Figure 1. The immigration process has the associated Hamiltonian

$$\alpha \left(a^+(a^-)^0 - (a^+)^0(a^-)^0\right) = \alpha\,(z-1)\,\Phi,$$

the emigration process operator is

$$\beta\left((a^+)^0 a^- - a^+a^-\right) = \beta(1-z)\frac{\partial}{\partial z},$$

and the fission operator is

$$\gamma\left((a^+)^2 a^- - a^+a^-\right) = \gamma(z^2-z)\frac{\partial}{\partial z}.$$

Combining these individual operators results in the associated master equation as is given in equation (2).

4.1 Method of characteristics

In order to use the method of characteristics (Zwillinger, 1992), rewrite the master equation as

$$\frac{\partial \Phi}{\partial t} + \left(-\gamma z^2 + (\beta+\gamma)z - \beta\right)\frac{\partial \Phi}{\partial z} = \alpha\,(z-1)\,\Phi. \tag{12}$$

Assume that there exists a differentiable parametrization $t = t(r,s)$ and $z = z(r,s)$ such that

$$\frac{\partial t}{\partial s} = 1, \tag{13}$$

$$t(r, s=0) = 0, \tag{14}$$

$$\frac{\partial z}{\partial s} = -\gamma z^2 + (\beta+\gamma)z - \beta, \tag{15}$$

$$z(r, s=0) = r, \tag{16}$$

$$\frac{\partial \Phi}{\partial s} = \alpha(z-1)\Phi, \tag{17}$$

$$\Phi(r, s=0) = 1 \cdot r^{\mu_0}. \tag{18}$$

Integrating (13) and using the initial condition (14) yields $t = s$. Integrating (15) and using the initial condition (16) yields

$$r = \frac{\beta(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)s}}{\gamma(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)s}}, \qquad \text{and} \qquad z = \frac{(\beta-\gamma r) + \beta(r-1)e^{(\beta-\gamma)s}}{(\beta-\gamma r) + \gamma(r-1)e^{(\beta-\gamma)s}}.$$

Lastly, integrating (17) and using the initial condition (18) yields the closed form expression

$$\Phi(t,z) = \left(\beta-\gamma\right)^{-\alpha/\gamma}\left[\frac{\beta(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)t}}{\gamma(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)t}}\right]^{\mu_0}\left[\frac{(\beta-\gamma)^2 e^{(\beta-\gamma)t}}{\gamma(z-1) + (\beta-\gamma z)e^{(\beta-\gamma)t}}\right]^{\alpha/\gamma}. \tag{19}$$

Using standard methods of analysis, it can be shown that the GF satisfies the essential properties

$$\lim_{z\to 1}\Phi(t,z) = 1, \qquad \text{and} \qquad \lim_{t\to 0}\Phi(t,z) = 1\cdot z^{\mu_0}$$

as well as the SE given in equation (12).

4.2 Expected value

We now show that the quantum/stochastic paradigm predicts the familiar ODE model found above. We find that $N\Phi$ is

$$-\frac{z\left(\frac{e^{t(\beta-\gamma)}(\beta-\gamma z)+\beta(z-1)}{e^{t(\beta-\gamma)}(\beta-\gamma z)+\gamma(z-1)}\right)^{\mu_0}\left(\frac{(\beta-\gamma)e^{t(\beta-\gamma)}}{e^{t(\beta-\gamma)}(\beta-\gamma z)+\gamma(z-1)}\right)^{\alpha/\gamma}\left(\mu_0(\beta-\gamma)^2 e^{t(\beta-\gamma)} + \alpha\left(e^{t(\beta-\gamma)}-1\right)\left(e^{t(\beta-\gamma)}(\beta-\gamma z)+\beta(z-1)\right)\right)}{\left(e^{t(\beta-\gamma)}(\beta-\gamma z)+\beta(z-1)\right)\left(\gamma + e^{t(\beta-\gamma)}(\gamma z-\beta)+\gamma(-z)\right)}.$$

Evaluating at $z = 1$ yields the expected value is

$$\mu(t) = \frac{\alpha - e^{t(\gamma-\beta)}(\alpha + \mu_0(\gamma-\beta))}{\beta-\gamma}, \tag{20}$$

with initial condition $\mu(0) = \mu_0$. Notice that the first moment $\mu(t)$ satisfies the standard ODE population model

$$\begin{aligned}\mu'(t) &= \alpha + \frac{(\gamma-\beta)e^{-t(\beta-\gamma)}\left(-\alpha + \alpha e^{t(\beta-\gamma)} + \beta\mu_0 - \gamma \mathrm{u}0\right)}{\beta-\gamma} \\ &= (\gamma-\beta)\mu + \alpha \end{aligned} \tag{21}$$

as is found in single species population dynamics. This result means that this standard ODE model can be derived via quantum mechanics instead of making macroscopic heuristic assumptions as is commonly done.

4.3 Long-term behavior of the individual probabilities

With $\alpha = \gamma = 1, \beta = 2$, and $\mu_0 = 1$. Collecting the coefficients of the monomials z^n yields the infinite system of first order ODE/difference equations

$$\begin{aligned}\phi_0'(t) &= -\phi_0(t) + 2\phi_1(t) \\ \phi_1'(t) &= \phi_0(t) - 4\phi_1(t) + 4\phi_2(t) \\ \phi_2'(t) &= 2\phi_1(t) - 7\phi_2(t) + 6\phi_3(t) \\ \phi_3'(t) &= 3\phi_2(t) - 10\phi_3(t) + 8\phi_4(t) \\ &\vdots \\ \phi_{n+1}'(t) &= (n+1)\phi_n(t) - (3n+4)\phi_{n+1}(t) + (2n+4)\phi_{n+2}(t) \qquad \text{for} \quad n \geq 0.\end{aligned}$$

Since the GF can be written as $\Phi(t,z) = \sum_{n=0}^{\infty}\phi_n(t)z^n$ and the $\{\phi_n(t)\}$ is a valid probability distribution, then $\sum_{n=0}^{\infty}\phi_n(t) = 1$ and so $\sum_{n=0}^{\infty}\phi_n'(t) = 0$. If a steady state exists for each density function then $\lim_{t\to\infty}\phi_j'(t) = 0$. The infinite system of recurrence ODEs reduces to the infinite system of difference equations

$$\begin{aligned}-\phi_0 + 2\phi_1 &= 0 \\ \phi_0 - 4\phi_1 + 4\phi_2 &= 0 \\ 2\phi_1 - 7\phi_2 + 6\phi_3 &= 0 \\ 3\phi_2 - 10\phi_3 + 8\phi_4 &= 0 \\ &\vdots \\ (n+1)\phi_n - (3n+4)\phi_{n+1} + (2n+4)\phi_{n+2} &= 0 \qquad \text{for} \quad n \geq 0.\end{aligned}$$

Using induction proves the desired result that $\lim_{t\to\infty}\phi_n(t) = (1/2)^{n+1}$ for $n \in \mathbb{N}$.

5 Discussion

In this paper we have two main results:

i. The standard immigration, emigration and fission ODE model was derived using quantum mechanic techniques, and

ii. the probability densities of having exactly n objects at time $t \geq 0$ are explicitly found and exhibit quantum tunneling.

The second result has major implications for models such as the susceptible-infectious-recovered (SIR) model. The process for deriving the associated Schrodinger equation is based on extending the GF to

$$\Phi(t,S,I,R) := \sum_{j,k,l}\phi_{j,k,l}(t)S^jI^kR^l.$$

The resulting calculations are similar to the above work however are very tedious. Preliminary results show that once an infection is introduced into a disease free population, then the infection persists, albeit at a low level. This suggests that the quantum approach introduces a natural reservoir for the pathogen.

In the modified quantum approach taken here, the solution to an appropriate Schrodinger equation $\Phi(t,z) = \sum_{n=0}^{\infty} \phi_n(t) z^n$ describes how the probability of having exactly n objects at time t and is given quantitatively by ϕ_n. The classical heuristic ODE approach yields the single, stable and non-negative equilibrium point $\bar{\mu} = \frac{\alpha}{\beta-\gamma}$ provided $\beta > \gamma$. For the specific choices $\alpha = \gamma = 1$, $\beta = 2$ and the initial condition $\mu_0 = 1$ yields $\bar{\mu} = 1$. In other words the classical system is approaching the stable state of one object. For these choices of parameters of the modified quantum approach, the long-term probabilities satisfy the condition that $\lim_{t\to\infty} \phi_n(t) = (1/2)^{n+1}$ for all $n \in \mathbb{N}$. This conflicts with the heuristic expectation that $\lim_{t\to\infty} \phi_1(t) = 1$ while for the modified quantum approach taken here it is actually $\phi_1(t) = 1/4$. The quantum tunneling of probabilities means that there is a probability, albeit small that there could be 100 objects as $t \longrightarrow \infty$.

The potential utility for quantum tunneling of probabilities of population dynamics is that future work will show that the standard SIR ODE model does not allow for the explanation of a natural reservoir of a pathogen unless heuristically assumed. The quantum approach taken here predicts the specific long-term probability of a pathogen reservoir.

Another direction of research is linking the SIR quantum model with gene regulatory networks (GRN). Preliminary results suggest that the SIR quantum approach is isomorphic with GRN and could be used to establish quantum tunneling effects and make quantitative predictions of defective transitions in a GRN. Lastly, since the process of differentiation is usually performed an integer number of times, this quantum approach could be extended to fractional calculus.

Conflict of Interest

The work has no potential conflict of interest.

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